Robot Arm Process
Modeling and Control

October 2017

Part I

Dynamic models of robots arms Euler-Lagrange approach

1. Euler-Lagrange equations

The **Lagrangian** is defined as

$$L = K - P, \tag{1}$$

where K represents the total kinetic energy of the system and P represents the total potential energy of the system.

The **Euler-Lagrange equations** that describes the dynamics of a $n - DOF$ mechanical system are [1] :

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \tau_i, \qquad i = 1, ..., n, \tag{2}$$

where q_i represent generalized coordinates (in our case the joint angles) and τ_i generalized forces (in our case motor torques) [2] .

The **matrix form** of the Euler-Lagrange equations is:

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau \tag{3}$$

where $q = [q_1, ..., q_n]^T, \tau = [\tau_1, ..., \tau_n]^T$.

The matrix $D(q)$ is called inertia matrix, it is symmetric and positive definite, and can be expressed in terms of the kinetic energy:

$$K = \frac{1}{2}\dot{q}^T D(q)\dot{q} = \frac{1}{2}\sum_{i,j}^{n} d_{i,j}(q)\dot{q}_i\dot{q}_j. \tag{4}$$

The matrix $C(q)$ takes into account centrifugal and Coriolis terms, and each $k, j - th$ matrix element can be calculated as:

$$c_{kj} = \frac{1}{2}\sum_{i=1}^{n} \underbrace{\left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right\}}_{c_{ijk}} \dot{q}_i. \tag{5}$$

The last term $G(q)$, sometimes called gravity term, is a column vector $G = [g_1 ... g_n]^T$, where each $k - th$ term is derived from the potential energy:

$$g_k(q) = \frac{\partial P}{\partial q_k}, \qquad k = 1, ..., n. \tag{6}$$

2. A 2DOF robot arm with spatial movement

Consider a 2DOF robot arm with two revolute joints, that can move in a 3D Cartesian space, with the schematic representation from Figure 1. Because the first rotation axis is on the X axis, an the second on the Y axis, that robot can move in a 3D space.

Geometric Model

The geometric model can be derived through transformation matrices from the base frame to the end effector frame. The base frame coincides with the first frame (that is the frame of joint 1, with origin O_1 in the center of the joint). Thus the transformation matrix T_{01} is simply a rotation around X:

$$T_{01} = Rot(x, q_1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(q_1) & -sin(q_1) & 0 \\ 0 & sin(q_1) & cos(q_1) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

[1] The Euler–Lagrange equations are also used in optimal control and calculus of variations. See [3] for an interesting discussion on the interplay between the physical interpretation and the mathematical insight.

[2] This presentation is based on [1]. For a formal derivation of the Euler-Lagrange equations from Newton's Laws based on the principle of virtual work see chapter 6.1 of the book.

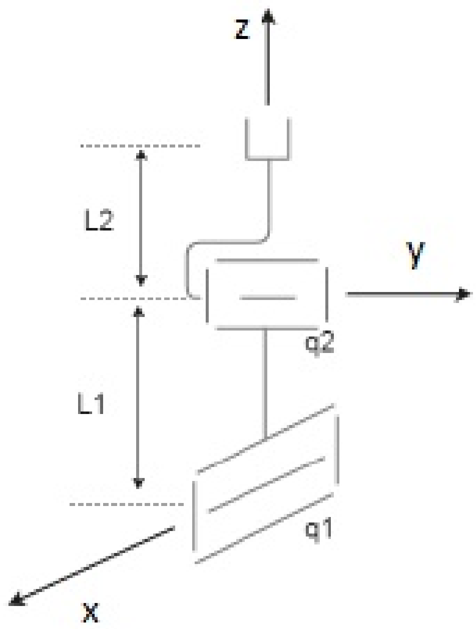

Figure 1: Schematic representation of a 2DOF robot arm

From Frame 1 we arrive at Frame 2 (corresponding to the joint 2) through a translation on Z and a rotation around Y (T_{12}):

$$T_{12} = Transl(z, L_1) \cdot Rot(y, q_2) = \begin{bmatrix} cos(q2) & 0 & sin(q2) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(q2) & 0 & cos(q2) & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Finally, the end effector frame is obtained through a translation on Z (T_{23}):

$$T_{23} = Transl(z, L_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The transformation matrix from the based frame to the end effector, that is the **geometric model** is obtained through multiplication:

$$T = T_{03} = T_{01} \cdot T_{12} \cdot T_{23} = \begin{bmatrix} cos(q_2) & 0 & sin(q_2) & L_2 sin(q_2) \\ sin(q_1)sin(q_2) & cos(q_1) & -cos(q_2)sin(q_1) & -sin(q_1)(L_1 + L_2 cos(q_2)) \\ -cos(q_1)sin(q_2) & sin(q_1) & cos(q_1)cos(q_2) & cos(q_1)(L_1 + L_2 cos(q_2)) \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (7)$$

The position of the end effector with respect to the joint angles q_1 and q_2 is given by the first three elements of the 4th column:

$$x = L_2 sin(q_1), \qquad y = L_1 sin(q_1) - L_2 sin(q_1)cos(q_2), \qquad z = L_1 cos(q_1) + L_2 cos(q_1)cos(q_2). \quad (8)$$

The orientation of the end effector is given by the submatrix R (lines 1-3 and columns 1-3 of T):

$$R = \begin{bmatrix} cos(q_2) & 0 & sin(q_2) \\ sin(q_1)sin(q_2) & cos(q_1) & -cos(q_2)sin(q_1) \\ -cos(q_1)sin(q_2) & sin(q_1) & cos(q_1)cos(q_2) \end{bmatrix}. \quad (9)$$

Jacobian

The Jacobian relates the joint velocities to the linear and angular velocities of the end effector [3]. We will write the Jacobian as:

$$J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} = [J_1, ..., J_i, ..., J_n], \tag{10}$$

where J_v is the Jacobian component for the linear velocities (v), J_ω the Jacobian component for the angular velocities (ω), and J_i is the Jacobian coresponding to each i^{th} joint.

If joint i is prismatic, with a translation movement along axis z, then

$$J_i = \begin{bmatrix} z_{i-1} \\ 0 \end{bmatrix}. \tag{11}$$

If joint i is revolute, with rotation axis z, then

$$J_i = \begin{bmatrix} z_{i-1} \times (O_n - O_{i-1}) \\ z_{i-1} \end{bmatrix}, \tag{12}$$

where '$\times$' stands for vector product.

Note that O_i is given by the first three elements of the 4^{th} column of T_{0i}, while z_i (respectively y_i,x_i) is given by column three of R_{0i} (respetively column two, column one).

In what follows, we will denote with $J_{vc,i}$ the Jacobian coresponding to linear velocities with respect to the center of mass of link i [4]. Because we consider the center of mass at the middle of each link, the only thing that changes in $J_{vc,i}$, compared to $J_{v,i}$ is that the length of link i is divided by two (that is L_i is replaced by $L_i/2$).

For the 2DOF robot arm from Figure 1, with the geometric model 7, the **Jacobian** is:

$$J = \begin{bmatrix} 0 & \frac{L_2}{2}cos(q_2) \\ -\frac{L_2}{2}cos(q_1)cos(q_2) - L_1cos(q_1) & \frac{L_2}{2}sin(q_1)sin(q_2) \\ -\frac{L_2}{2}sin(q_1)cos(q_2) - L_1sin(q_1) & -\frac{L_2}{2}cos(q_1)sin(q_2) \\ 1 & 0 \\ 0 & cos(q_1) \\ 0 & sin(q_1) \end{bmatrix}. \tag{13}$$

Thus, if we refer to link 2, the angular and linear Jacobians are:

$$J_{vc2} = J_{vc} = \begin{bmatrix} 0 & \frac{L_2}{2}cos(q_2) \\ -\frac{L_2}{2}cos(q_1)cos(q_2) - L_1cos(q_1) & \frac{L_2}{2}sin(q_1)sin(q_2) \\ -\frac{L_2}{2}sin(q_1)cos(q_2) - L_1sin(q_1) & -\frac{L_2}{2}cos(q_1)sin(q_2) \end{bmatrix}, \tag{14}$$

$$J_{\omega 2} = J_\omega = \begin{bmatrix} 1 & 0 \\ 0 & cos(q_1) \\ 0 & sin(q_1) \end{bmatrix}. \tag{15}$$

Further on, the angular and linear Jacobians for link 1 [5] can be determined as:

$$J_{vc1} = \begin{bmatrix} 0 & 0 \\ -\frac{L_1}{2}cos(q_1) & 0 \\ -\frac{L_1}{2}sin(q_1) & 0 \end{bmatrix}, \tag{16}$$

$$J_{\omega 1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{17}$$

[3] The Jacobian presented here is sometimes reffered in the literature as geometric Jacobian, to distinguish it from the analytical Jacobian. It is often used in practice to calculate the Jacobian in respect with a given point (center of gravity) in 3D space. For a detailed discussion on the Jacobian and velocity kinematics see [1] - chapter 4.

[4] Angular velocity is not a propriety of individual points, but linear velocity can be.

[5] Reconsider the calculations as if link 2 does not exist.

Lagrangian

The Lagrangian is composed out of kinetic energy and potential energy. The **kinetic energy** has a translational and a rotational component

$$K = K_{transl} + K_{rot}, \tag{18}$$

given by the expressions:

$$K_{transl} = \frac{1}{2}m_1 v_{c1}^T v_{c1} + \frac{1}{2}m_2 v_{c2}^T v_{c2} = \frac{1}{2}\dot{q}^T(m_1 J_{vc1}^T J_{vc1} + m_2 J_{vc2}^T J_{vc2})\dot{q}, \tag{19}$$

and

$$K_{rot} = \frac{1}{2}\dot{q}^T(J_{\omega 2}^T R_2 I_2 R_2^T J_{\omega 2} + J_{\omega 1}^T R_1 I_1 R_1^T J_{\omega 1})\dot{q}, \tag{20}$$

with[6]

$$R_2 = R, \quad R_1 = T_{01}(1:3, 1:3), I_2 = diag\{0, I_{2y}, 0\}, \quad I_1 = diag\{I_{1x}, 0, 0\}. \tag{21}$$

After calculating the expressions for both components of the kinetic energy, we obtain the inertia matrix $D(q)$ as

$$D(q) = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} = \begin{bmatrix} I_{1x} + \frac{L_1^2 m_1}{4} + L_1^2 m_2 + \frac{L_2^2 m_2}{4} cos^2(q_2) + L_1 L_2 m_2 cos(q_2) & 0 \\ 0 & \frac{m_2 L_2^2}{4} + I_{2y} \end{bmatrix}. \tag{22}$$

In deriving matrix $C(q, \dot{q})$, we first calculate each c_{ijk} term from (5):

$$\begin{aligned}
c_{111} &= \frac{\partial d_{11}}{\partial q_1} + \frac{\partial d_{11}}{\partial q_1} - \frac{\partial d_{11}}{\partial q_1} = 0, \\
c_{112} &= \frac{\partial d_{21}}{\partial q_1} + \frac{\partial d_{21}}{\partial q_1} - \frac{\partial d_{11}}{\partial q_2} = \frac{L_2^2 m_2}{4} sin(2q_2) + L_1 L_2 m_2 sin(q_2), \\
c_{121} &= \frac{\partial d_{12}}{\partial q_1} + \frac{\partial d_{11}}{\partial q_2} - \frac{\partial d_{12}}{\partial q_1} = -\frac{L_2^2 m_2}{4} sin(2q_2) - L_1 L_2 m_2 sin(q_2), \\
c_{122} &= \frac{\partial d_{22}}{\partial q_1} + \frac{\partial d_{21}}{\partial q_2} - \frac{\partial d_{12}}{\partial q_2} = 0, \\
c_{211} &= \frac{\partial d_{11}}{\partial q_2} + \frac{\partial d_{12}}{\partial q_1} - \frac{\partial d_{21}}{\partial q_1} = c_{121}, \\
c_{212} &= \frac{\partial d_{21}}{\partial q_2} + \frac{\partial d_{22}}{\partial q_1} - \frac{\partial d_{21}}{\partial q_2} = 0, \\
c_{221} &= \frac{\partial d_{12}}{\partial q_2} + \frac{\partial d_{12}}{\partial q_2} - \frac{\partial d_{22}}{\partial q_1} = 0, \\
c_{222} &= \frac{\partial d_{22}}{\partial q_2} + \frac{\partial d_{22}}{\partial q_2} - \frac{\partial d_{22}}{\partial q_2} = 0.
\end{aligned}$$

In the end we obtain the matrix:

$$C(q, \dot{q}) = \begin{bmatrix} -\frac{L_2^2 m_2}{8} sin(2q_2)\dot{q}_2 - \frac{1}{2} L_1 L_2 m_2 sin(q_2)\dot{q}_2 & -\frac{L_2^2 m_2}{8} sin(2q_2)\dot{q}_1 - \frac{1}{2} L_1 L_2 m_2 sin(q_2)\dot{q}_1 \\ \frac{L_2^2 m_2}{8} sin(2q_2)\dot{q}_1 + \frac{1}{2} L_1 L_2 m_2 sin(q_2)\dot{q}_1 & 0 \end{bmatrix}. \tag{23}$$

The **potential energy** is determined by multiplying the mass by the gravitational acceleration and the height at the center of mass:

$$P_1 = m_1 g \frac{L_1}{2} cos(q_1), \qquad P_2 = m_2 g \Big(L_1 cos(q_1) + \frac{L_2}{2} cos(q_1) cos(q_2) \Big), \qquad P = P_1 + P_2. \tag{24}$$

Based on (6), the gravity term is determined as:

$$G(q) = \begin{bmatrix} -\frac{m_1 g L_1 + 2 m_2 g L_1}{2} sin(q_1) - \frac{m_2 g L_2}{2} sin(q_1) cos(q_2) \\ -\frac{m_2 g L_2}{2} cos(q_1) sin(q_2) \end{bmatrix}. \tag{25}$$

This completes the dynamic model for our robot arm.

[6]Because in practice the off diagonals terms of the inertia matrices are neglectable, we consider here only the Principle Moments of Inertia corresponding to each rotation axis (I_x, I_y or I_z). Note that the inertia moments are expressed in respect with the body attached frame.

3. Actuator dynamics

Each joint of the robot arm is controlled by an electric motor. The model of a armature controlled DC motor is ([4]) [7] :

$$J_{mi}\ddot{\theta}_i + \underbrace{B_i\dot{\theta}_i}_{damping+backEMF} + \underbrace{F_{mi}}_{friction} + \underbrace{r_i\tau_i}_{load\,torque} = \underbrace{(K_{mi}/R_{ai})v_i}_{applied\,torque}, \quad i = 1, 2, \tag{26}$$

where θ_i is the rotor angular position of the i^{th} motor, v_i is the control voltage, J_{mi} is the motor inertia, F_{mi} denotes friction, R_{ai} is the armature resistance, K_{mi} is the torque constant. r_i is the gear ratio such that:

$$q_i = r_i\theta_i. \tag{27}$$

Finally $B_i = B_{mi} + K_{bi}K_{mi}/R_{ai}$, where B_{mi} is the damping constant and K_{bi} is the back EMF constant.

In matrix form, the motor equations can be written all together as:

$$J_m\ddot{\theta} + B\dot{\theta} + F_m + R\tau = K_m v, \tag{28}$$

where

$$J_m = diag\{J_{mi}\}, \quad B = diag\{B_i\}, \quad R = diag\{r_i\}, F_m = diag\{F_{mi}\}$$

$$K_m = diag\{K_{mi}/R_{ai}\}, v = [v_1\, v_2]^T, \theta = [\theta_1\, \theta_2]^T, \tau = [\tau_1\, \tau_2]^T.$$

4. Nonlinear dynamics of the robot process

The dynamic model 3 of a mechanical system so far includes inertia and gravity, Coriolis and centripetal forces. In practice other forces may need to be taken into account, like friction, backlash or elastic forces. For our present study, we will consider sufficient to add only a viscous friction term [8]

$$F(\dot{q}) = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \dot{q} = F_b\dot{q}. \tag{29}$$

The robot dynamic model now becomes:

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + F(\dot{q}) + G(q) = \tau. \tag{30}$$

If we use (27) to replace θ with q in (28), isolate τ and plugin in the expression in 30 we obtain

$$(J_m + R^2D(q))\ddot{q} + (B + R^2C(q, \dot{q}))\dot{q} + (RF_m + R^2F(\dot{q})) + R^2G(q) = RK_m v, \tag{31}$$

which can be rewritten more compactly, by a proper change of notations, as:

$$D'(q)\ddot{q} + C'(q, \dot{q})\dot{q} + F'(\dot{q}) + G'(q) = K'v. \tag{32}$$

This is now the nonlinear model of our robot process that we want to control, which has as inputs motor voltages and outputs joint variables [9].

[7] The armature inductance L_{ai} is considered negligible.

[8] Viscous friction is also called dynamic friction. Although static friction is very important in practice, it is usually ignored in the analysis and design phase, because it is difficult to model exactly, and it would complicate the controller design.

[9] The sensor dynamics is considered negligible.