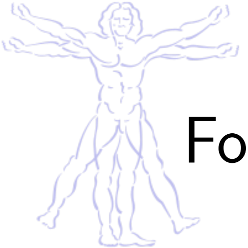


Forward Dynamics

Making things move

UNIVERSITATEA BABEȘ-BOLYAI

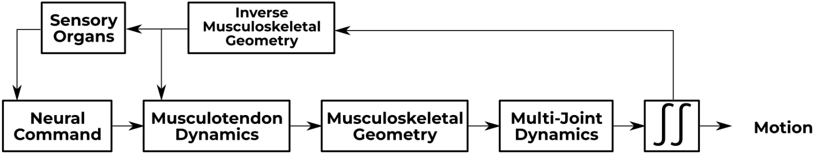
Last update: May 10, 2021

Agenda

-
-

Musculoskeletal modelling

Block diagram



Dynamic modeling

What is it all about?

Kinematics:

Dynamics (Kinetics):

Dynamic modeling

What is it all about?

Kinematics: description of motion of bodies or system of bodies

Dynamics (Kinetics):

Dynamic modeling

What is it all about?

Kinematics: description of motion of bodies or system of bodies

Dynamics (Kinetics): description of the causes resulting in those motions (i.e. forces and torques)

Dynamic modeling

What is it all about?

Dynamic model

A set of equations that gives us the relationship between input joint forces/torques and resulting joint accelerations.

Dynamic modeling

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Why is this useful?

Dynamic modeling

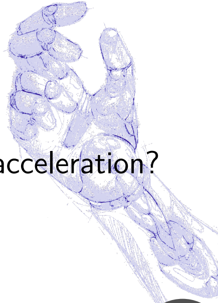
What is it all about?

Dynamic model

A set of equations that gives us the relationship between input joint forces/torques and resulting joint accelerations.

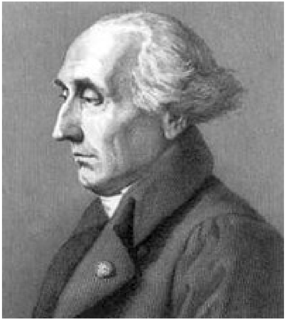
Why is this useful?

Do you know of any equation that relates force with acceleration?



Dynamic modeling

Lagrange-Euler formulation of mechanics



Between 1772 and 1788, Lagrange formulated mechanics in a more general way, more suitable for (bio-)mechanics later on.

Lagrangian mechanics

A more sophisticated formulation of mechanics

Lagrange defined a basic quantity for any system of bodies as the difference between its kinetic and potential energy.

$$L = K - P$$

We call this quantity the Lagrangian of the system.

Lagrangian mechanics

A more sophisticated formulation of mechanics

Using this quantity, we can describe the evolution of any system of bodies under the influence of a set of external forces/torques using the following equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = \tau$$

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Where y are some "generalized coordinates"

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Where y are some “generalized coordinates”
τ should be in the same coordinates.

Definitions

Potential energy

Reference for potential

Potential is only important when considering the difference of potential. Therefore, the reference is not important, as long as it does not change over time, and we use the same one for all the objects.

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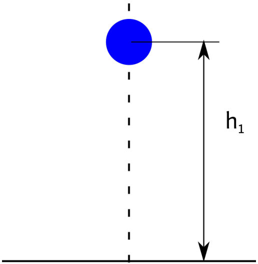

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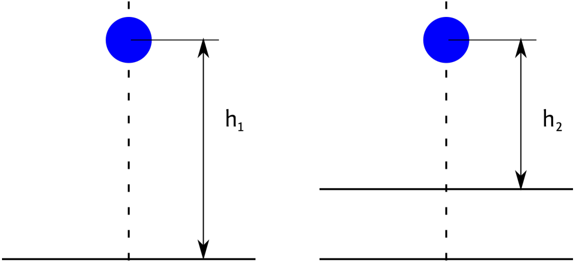

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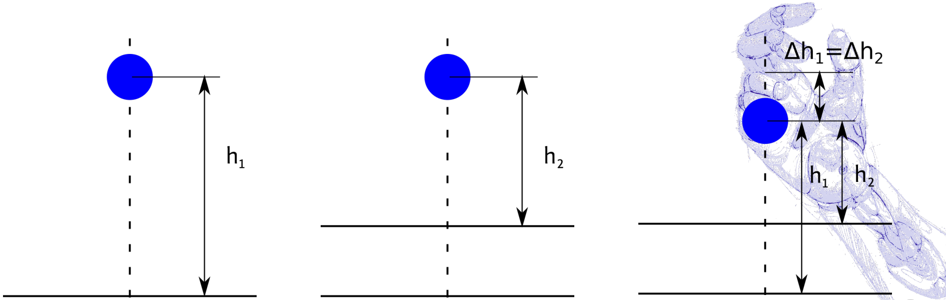
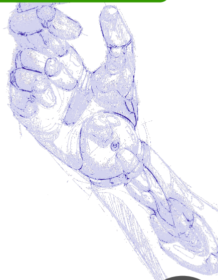


Definitions

Kinetic energy

Kinetic energy

The energy of an object that it possesses due to its motion.



Definitions

Kinetic energy

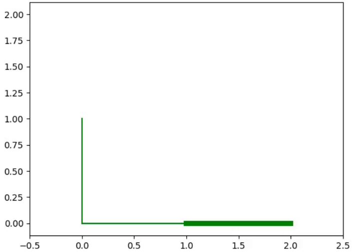
Kinetic energy

The energy of an object that it possesses due to its motion.

What properties are influencing the kinetic energy of an object.

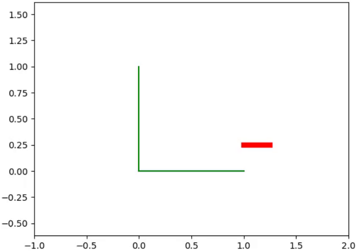
Definitions

Kinetic energy



Kinetic energy due to

linear velocity

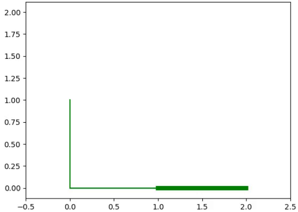


Kinetic energy due to

angular velocity

Definitions

Kinetic energy



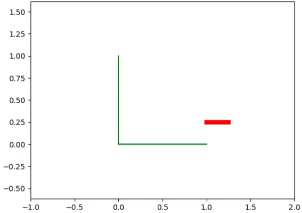
The equation of kinetic energy due to linear velocity is:

$$K_{linear} = \frac{1}{2} m u^2$$

Where m is the mass of the object and u is the magnitude of its velocity (i.e. regardless of direction).

Definitions

Kinetic energy



The equation of kinetic energy due to angular velocity is:

$$K_{angular} = \frac{1}{2} I \omega^2$$

Where I is the moment of inertia of the object, and ω is its angular velocity.

Definitions

Kinetic energy

Total kinetic energy

The total kinetic energy of an object is the sum of its linear and angular kinetic energy.

$$K_{total} = K_{linear} + K_{angular} = \frac{1}{2}(mu^2 + I\omega^2)$$

Definitions

Moment of inertia

The moment of inertia shows us how 'difficult' is it to rotate an object around an arbitrary axis. It is related with how the mass of the object is distributed in space.

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

This 'difficulty' might be different for the same object, but different axes.

Definitions

Kinetic energy

Let's have a look at the angular kinetic energy again. We saw that:

$$K_{angular} = \frac{1}{2} I \omega^2$$

Definitions

Kinetic energy

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$$K_{angular} = \frac{1}{2} I \omega^2$$

But if I, is a tensor and ω a scalar, then the kinetic energy will be a tensor as well.

Definitions

Kinetic energy

But Kinetic energy is a scalar, and the angular velocity is a vector.

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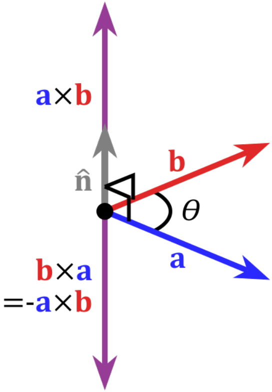
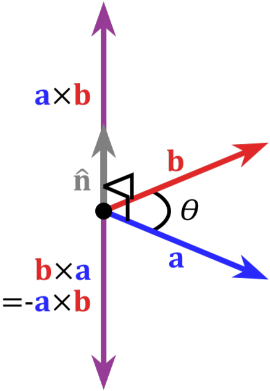


Definitions

Kinetic energy

But Kinetic energy is a scalar, and the angular velocity is a vector. Therefore, we can calculate the angular kinetic energy using the vectorial equation:



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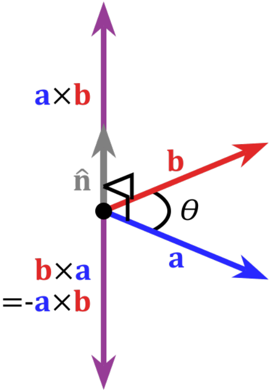


$$K_{angular} = \frac{1}{2}\omega^T I \omega$$

Definitions

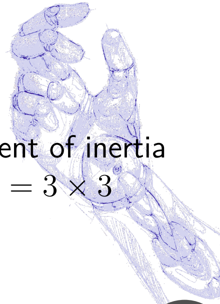
Bringing it all together

We define the Lagrangian as the difference between Kinetic and Potential energy of our system

$$L = K - P$$

where:

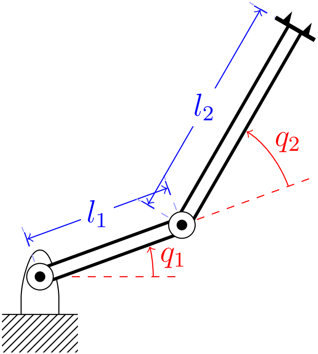
Potential Energy	Kinetic Energy	Moment of inertia
$P = mgh$	$K = \frac{1}{2}(mu^2 + \omega^T I \omega)$	$I = 3 \times 3$



Lagrangian of a mechanism

How do we calculate it?

Let's take an 'easy' example of a 2-link planar mechanism.



Let's assume that segments have masses m_1 and m_2 respectively.

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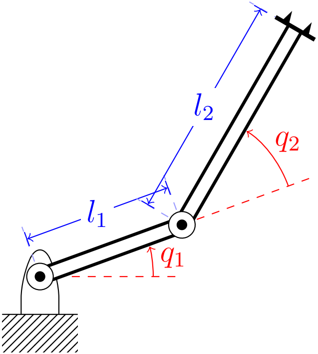

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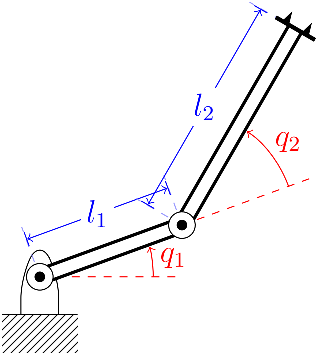


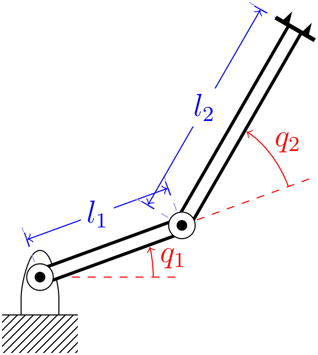
Let's assume that segments have masses m_1 and m_2 respectively.

We need to calculate its Lagrangian in terms of some 'generalized' coordinates.

Lagrangian of a mechanism

How do we calculate it?

Which 'generalized' coordinates are most convenient?



Lagrangian of a mechanism

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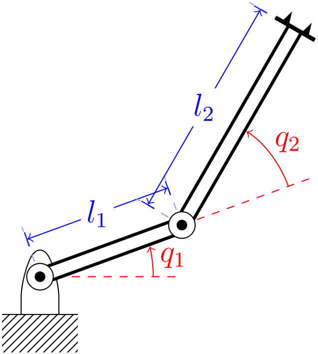

X, Y?

Lagrangian of a mechanism

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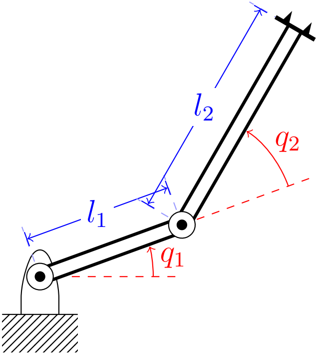

X, Y?
θ, r?

Lagrangian of a mechanism

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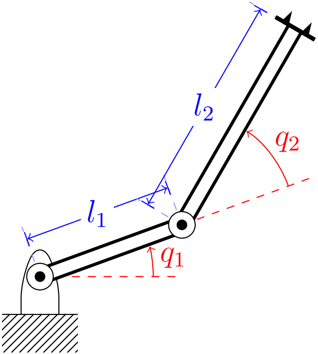

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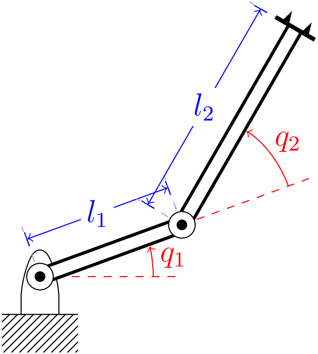
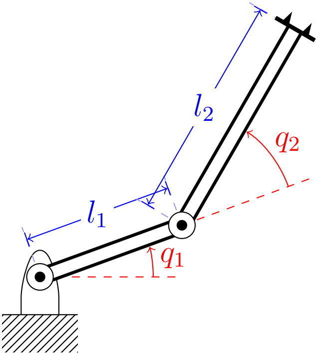


X, Y?
θ, r?
$q1$, $q2$?
Why?

Lagrangian of a mechanism

Potential energy

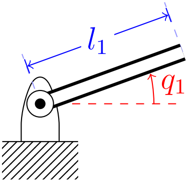
We need to calculate the total dynamic energy of the system with respect to $q, \dot{q}$.



The total dynamic energy is the sum of the dynamic energies of each segment. What is the dynamic energy of each segment?

Lagrangian of a mechanism

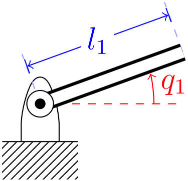
Potential energy



We consider the mass of the link to be concentrated at its center of mass.

Lagrangian of a mechanism

Potential energy

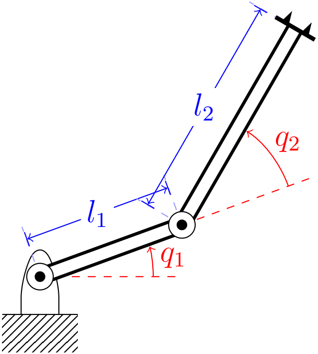


We consider the mass of the link to be concentrated at its center of mass. Therefore:

$$P_1(q, \dot{q}) = m_1 g \frac{l_1}{2} sin q_1$$

Lagrangian of a mechanism

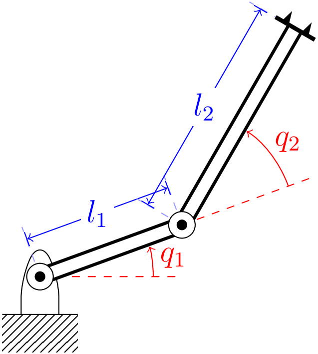
Potential energy



For the second segment, we think alike:

Lagrangian of a mechanism

Potential energy



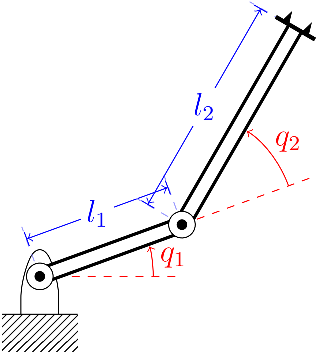
For the second segment, we think alike:

$P_2(q, \dot{q}) =$

$$m_2 g \left(l_1 sin q_1 + \frac{l_2}{2} sin(q_1 + q_2) \right)$$

Lagrangian of a mechanism

Potential energy



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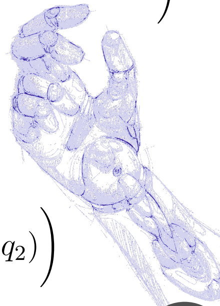

The total potential energy is therefore:

$$P(q,\dot{q}) = m_1 g \frac{l_1}{2} sin q_1 + m_2 g \left(l_1 sin q_1 + \frac{l_2}{2} sin(q_1 + q_2) \right)$$

Lagrangian of a mechanism

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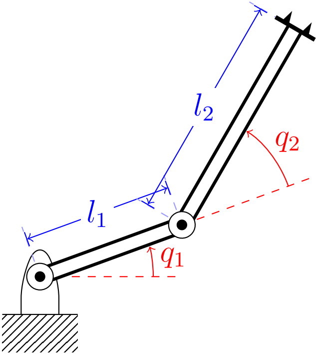

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Does P depend on $\dot{q}$?

Lagrangian of a mechanism

Kinetic energy

Once again, we take the kinetic energy of each segment with respect to $q, \dot{q}$ and add them together.

$K_{total}(q, \dot{q}) = K_1(q, \dot{q}) + K_2(q, \dot{q})$

Lagrangian of a mechanism

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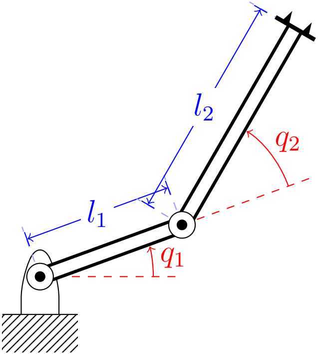
Let's start with the linear kinetic energy first.



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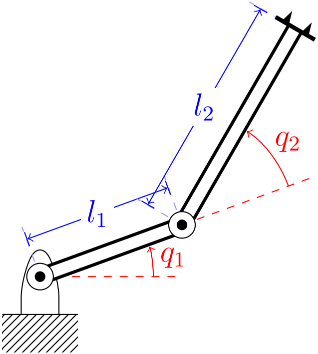
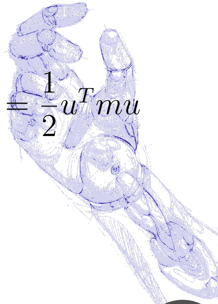


$$K_{lin}(q, \dot{q}) = \frac{1}{2} m u^2 = \frac{1}{2} u^T m u$$



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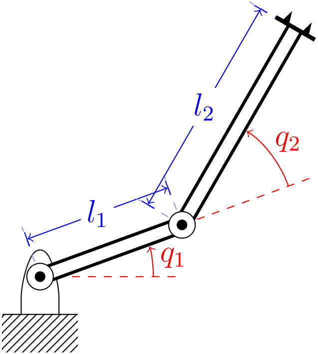

$$K_{lin}(q, \dot{q}) = \frac{1}{2} m u^2 = \frac{1}{2} u^T m u$$

Do we know what is u in terms of $q, \dot{q}$?

Lagrangian of a mechanism

Going back in time

How do we convert the joint velocity ($\dot{q}$) into linear velocity (u)?

Lagrangian of a mechanism

Going back in time

How do we convert the joint velocity ($\dot{q}$) into linear velocity (u)?

The jacobian!

$$u = J_u \dot{q}$$

What is the Jacobian?

Lagrangian of a mechanism

Going back in time

How do we convert the joint velocity ($\dot{q}$) into linear velocity (u)?

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What is the Jacobian?
Part 1, Part 2

Lagrangian of a mechanism

Kinetic energy

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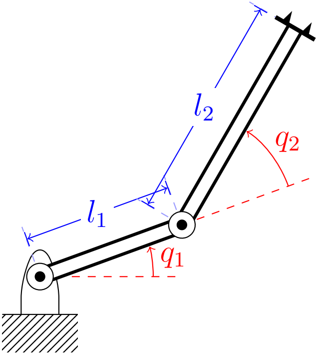

$$K_{lin}(q, \dot{q}) = \frac{1}{2}mu^2 = \frac{1}{2}\dot{q}^T \sum_{i=1}^{n}[J_{ui}^T m_i J_{ui}]\dot{q}$$

Lagrangian of a mechanism

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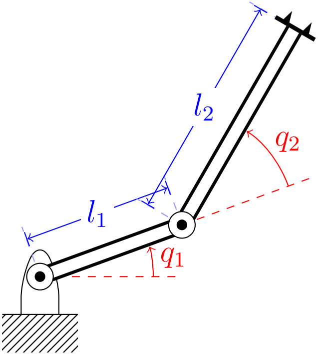

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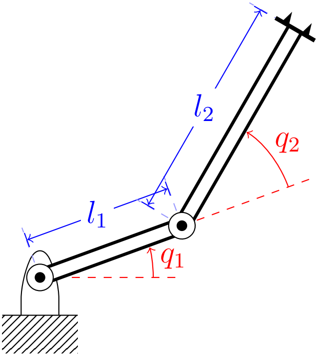
Remember $(AB)^T = B^T A^T$

Lagrangian of a mechanism

Kinetic energy

... and then the angular kinetic energy:



$$K_{ang}(q, \dot{q}) = \frac{1}{2}\omega^T I \omega$$

Lagrangian of a mechanism

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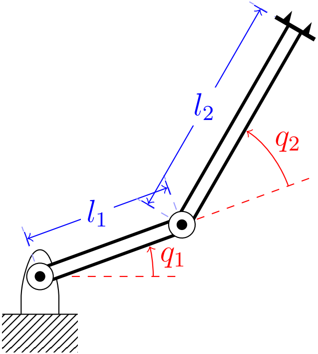

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I_i is expressed on the coordinate frame of link i

Lagrangian of a mechanism

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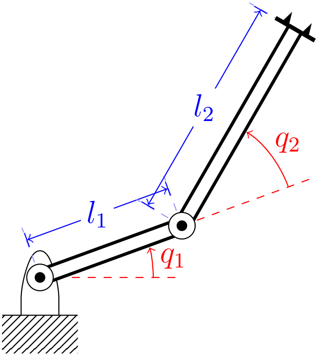

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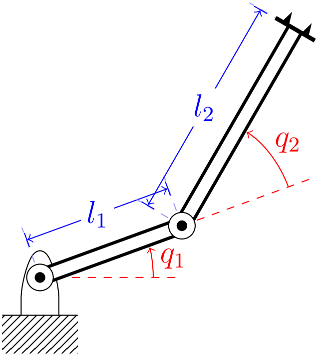
I_i is expressed on the coordinate frame of link i
But must be 'transformed' in the fixed coordinate frame

... and then the angular kinetic energy:



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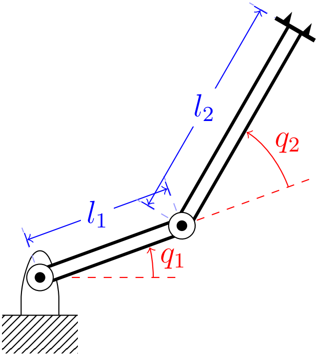
$I_i^o = R_i I_i^i R_i^T$

Lagrangian of a mechanism

Kinetic energy

... and then the angular kinetic energy:

$$K_{ang}(q,\dot{q}) = \frac{1}{2}\omega^T I \omega$$



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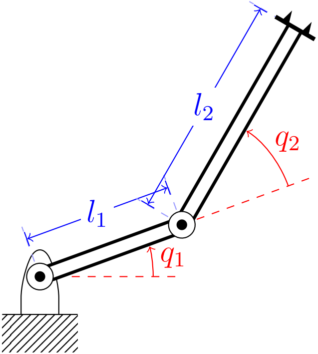
Therefore

$$K_{ang}(q,\dot{q}) = \frac{1}{2}\dot{q}^T \sum_{i=1}^{n} [J_{\omega i}^T R_i I_i^i R_i^T J_{\omega i}]\dot{q}$$

Lagrangian of a mechanism

Kinetic energy

Therefore, the total Kinetic energy of the mechanism is:



$$K(q, \dot{q}) = \frac{1}{2}\dot{q}^T \sum_{i=1}^{n} \left[J_{vi}^T m_i J_{vi} + J_{\omega i}^T R_i I_i R_i^T J_{\omega i} \right] \dot{q}$$

Lagrangian of a mechanism

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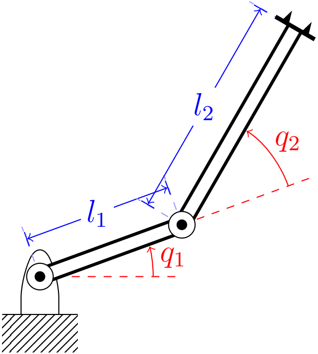

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Does K depend on $\dot{q}$?

Lagrangian of a mechanism

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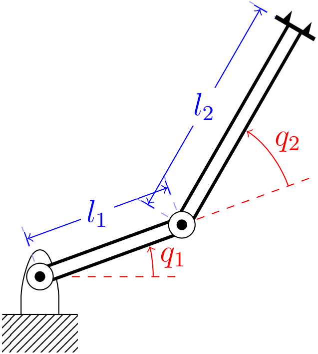

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Does K depend on $\dot{q}$? Does it depend on q?

Lagrangian of a mechanism

Let's plug it in the Lagrangian equation of motion

Eventually, we can write the kinetic energy in a condensed format:

$$K(q, \dot{q}) = \frac{1}{2}\dot{q}^T D(q)\dot{q}$$

And the potential energy:

$$P(q) = \sum_{i=1}^{n} g h_i(q) m_i$$

Therefore, the total Lagrangian is:

$$L(q, \dot{q}) = K - P = \frac{1}{2}\dot{q}^T D(q)\dot{q} - g\sum_{i=1}^{n} h_i(q) m_i$$

Lagrangian of a mechanism

Let's plug it all together

If we expand the first term, we get:

$$L = K - P = \frac{1}{2}\dot{q}^T D(q)\dot{q} - g\sum_{i=1}^{n} h_i(q)m_i$$

$$L = \frac{1}{2}\sum_{i,j}^{n} d_{ij}(q)\dot{q}_i\dot{q}_j - g\sum_{i=1}^{n} h_i(q)m_i$$

Lagrangian of a mechanism

Let's plug it all together

The equation of motion is:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau$$

Lagrangian of a mechanism

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$$\frac{\partial L}{\partial \dot{q}_k} = \sum_j d_{kj}\dot{q}_j$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \sum_j d_{kj}\ddot{q}_j + \sum_j \frac{d}{dt} d_{kj}\dot{q}_j = \sum_j d_{kj}\ddot{q}_j + \sum_{i,j} \frac{\partial d_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j$$

Lagrangian of a mechanism

Let's plug it all together

The second term is:

$$\frac{\partial L}{\partial q_k} = \frac{1}{2}\sum_{i,j}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial P}{\partial q_k}$$

Therefore, everything together is:

$$\sum_j d_{kj}\ddot{q}_j + \sum_{i,j}\{\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\}\dot{q}_i\dot{q}_j + \frac{\partial P}{\partial q_k} = \tau_k$$

Lagrangian of a mechanism

Condensed form

We can write this equation in a more general form:

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau$$

Lagrangian of a mechanism

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The matrix C has elements related to the **centrifugal** and **Coriolis** terms

Finally, the term g contains the dependence of the potential energy from the position of the mechanism.

Lagrangian of a mechanism

Christoffel symbols

The k, j-th element of matrix $C(q, \dot{q})$ is defined as:

$$c_{kj} = \sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i$$

$$= \sum_{i=1}^{n} \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i$$

Dynamic model

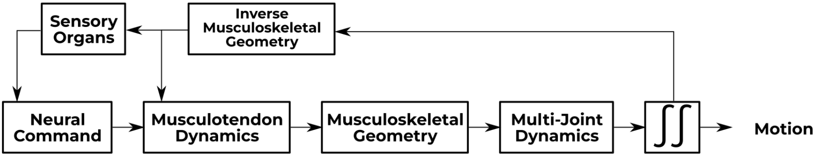
Torques

How do we calculate torques?

Dynamic model

Torques

How do we calculate torques?



Dynamic model

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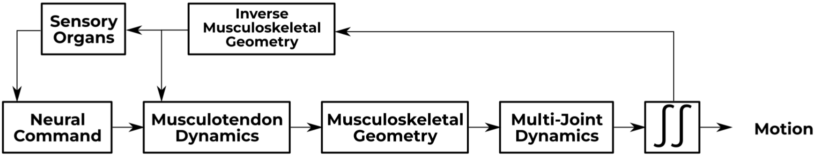

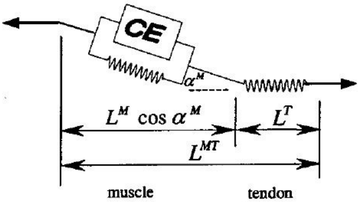

$$f_{iso}(\alpha(t) f_{AL}(l^M) f_v(i^M) + f_{PL}(l^M)) cosa - f_{iso} f_{SE}(l^T) = 0$$

Dynamic model

Forward dynamics

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

Dynamic model

Forward dynamics

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

$$\ddot{q} = D(q)^{-1}[\tau - C(q,\dot{q})\dot{q} - g(q)]$$

Dynamic model

Forward dynamics

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

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$$\ddot{q} = D(q)^{-1}[\tau(\alpha, l, \dot{l}) - C(q,\dot{q})\dot{q} - g(q)]$$

Inverse dynamics

Again the inverse?

How do we calculate the inverse dynamics?

Questions?