Inverse kinematics

From kinematics to joints

UNIVERSITATEA BABEȘ-BOLYAI

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Agenda

- Background
- Better understanding of the FK
- What is inverse kinematics?
- Solving
- Examples

Forward Kinematics

Better understanding

Definition

A transformation matrix that calculates the pose of the segment of interest in terms of the joint coordinates $q_1, q_2, \ldots, q_n$

$$\left[\begin{array}{ccc|c} X_X & Y_X & Z_X & P_x \\ X_Y & Y_Y & Z_Y & P_y \\ X_Z & Y_Z & Z_Z & P_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Forward Kinematics

Better understanding

This is basically a function of $q = [q_1, q_2, \ldots, q_n]$ that returns the pose of the end-effector

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$$f(q) \mapsto P_x, P_y, P_z, R$$

Forward Kinematics

Better understanding

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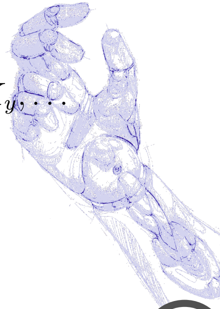
$$f(q) \mapsto P_x, P_y, P_z, R$$

Where R is the orientation defined in terms of $X_x, X_y, \dots$



Inverse Kinematics

Definition

Inverse kinematics is the 'inverse' of the forward kinematics

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$$g(P_x, P_y, P_z, R) \mapsto q = [q_1, q_2, \ldots, q_n]$$

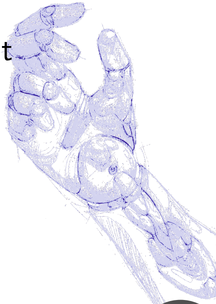
Inverse Kinematics

Definition

Inverse kinematics is the 'inverse' of the forward kinematics

$$g(P_x, P_y, P_z, R) \mapsto q = [q_1, q_2, \ldots, q_n]$$

A function that given a specific pose, returns the joint coordinates.



Inverse and Forward kinematics

What is the difference?

Forward kinematics

I want to know where will my end-effector be, if I give specific coordinates (values) to each joint

Inverse and Forward kinematics

What is the difference?

Forward kinematics

I want to know where will my end-effector be, if I give specific coordinates (values) to each joint

Inverse geometric model

I want to know what should the joint coordinates (values) be in order for my end-effector to reach a specific pose

Inverse and Forward kinematics

What is the difference?

The inverse model is usually more useful.

Inverse and Forward kinematics

What is the difference?

The inverse model is usually more useful.

Can you imagine why?

Inverse and Forward kinematics

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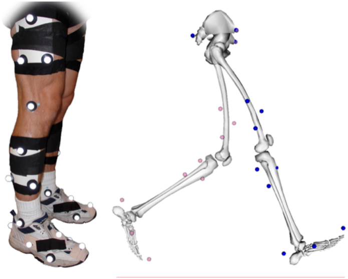

Inverse and Forward kinematics

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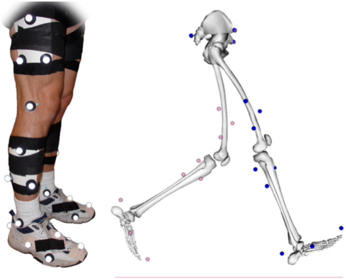

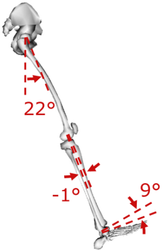

Inverse and Forward kinematics

What is the difference?

The inverse model is usually more useful.

But it is also most difficult to derive and we need the forward kinematics to derive it.

Inverse and Forward kinematics

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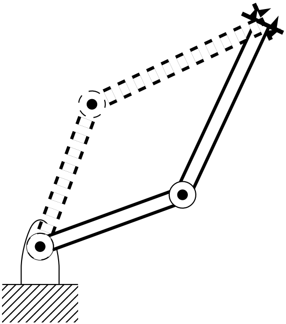
Can you imagine why?

Inverse kinematics model

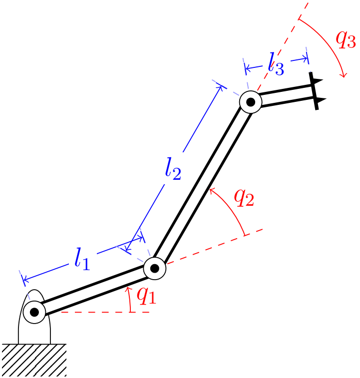
Why so difficult?



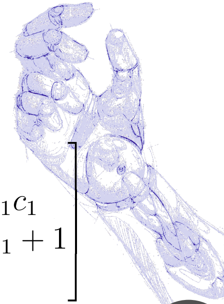
The inverse kinematic model might have more than one solution for a specific pose

Inverse kinematics model

Why so difficult?



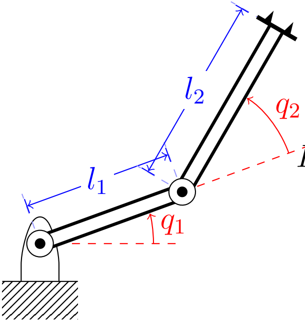
$$R_0^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{1,2,3} & -s_{1,2,3} & l_2 c_{1,2} + l_3 c_{1,2,3} + l_1 c_1 \\ 0 & s_{1,2,3} & c_{1,2,3} & l_2 s_{1,2} + l_3 s_{1,2,3} + l_1 s_1 + 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Inverse kinematics model

Derivation

The inverse model can be difficult to solve even for simple models



$$R(q_1, q_2) = \begin{bmatrix} c_{1,2} & -s_{1,2} & 0 & l_2 c_{1,2} + l_1 c_1 \\ s_{1,2} & c_{1,2} & 0 & l_2 s_{1,2} + l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse kinematics model

Derivation

$$\begin{bmatrix} c_{1,2} & -s_{1,2} & 0 & l_2 c_{1,2} + l_1 c_1 \\ s_{1,2} & c_{1,2} & 0 & l_2 s_{1,2} + l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} X_X & Y_X & Z_X & P_x \\ X_Y & Y_Y & Z_Y & P_y \\ X_Z & Y_Z & Z_Z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse kinematics model

Derivation

$$\begin{bmatrix} c_{1,2} & -s_{1,2} & 0 & l_2c_{1,2}+l_1c_1 \\ s_{1,2} & c_{1,2} & 0 & l_2s_{1,2}+l_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} X_X & Y_X & Z_X & P_x \\ X_Y & Y_Y & Z_Y & P_y \\ X_Z & Y_Z & Z_Z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$cos(q_1+q_2) = X_x = Y_y$$

$$sin(q_1+q_2) = X_y = -Y_x$$

$$l_2cos(q_1+q_2) + l_1cosq_1 = P_x$$

$$l_2sin(q_1+q_2) + l_1sinq_1 = P_y$$

$$0 = P_z$$

Inverse kinematics model

Derivation

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$$0 = P_z$$

How do we solve this?

Inverse kinematics model

Derivation

We are looking for this

$$q_1 = f(P_x, P_y)$$

$$q_2 = g(P_x, P_y)$$

Inverse kinematics model

Derivation

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$$q_1 = f(P_x, P_y)$$
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- Analytical solutions

Inverse kinematics model

Derivation

We are looking for this

$$q_1 = f(P_x, P_y)$$
$$q_2 = g(P_x, P_y)$$

- Analytical solutions
- Geometrical solutions

Inverse kinematics model

Derivation

We are looking for this

$$q_1 = f(P_x, P_y)$$
$$q_2 = g(P_x, P_y)$$

- Analytical solutions
- Geometrical solutions
- Numerical solutions

Inverse kinematics model

Analytical solutions

The idea is to try to find an equation for each joint variable q_n that only depends on the pose or on other joint variables that have already been expressed in terms of pose.

Inverse kinematics model

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- We equate the FK with the general homogeneous matrix

Inverse kinematics model

Analytical solutions

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- We identify joint variables that can be isolated

Inverse kinematics model

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- We equate the FK with the general homogeneous matrix
- We identify joint variables that can be isolated
- We identify pair of joint variables that can be simplified by division

Inverse kinematics model

Analytical solutions

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- We equate the FK with the general homogeneous matrix
- We identify joint variables that can be isolated
- We identify pair of joint variables that can be simplified by division
- We identify pair of joint variables that can be simplified by trigonometry

Inverse kinematics model

Analytical solutions

If not all joint variables are expressed as a function of the pose, we multiply from left(right) the inverse transformation of the first(last) joint.

Inverse kinematics model

Analytical solutions

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$$R_0^n = R_0^1 R_1^2 \dots R_{n-1}^n = R_g$$

Inverse kinematics model

Analytical solutions

If not all joint variables are expressed as a function of the pose, we multiply from left(right) the inverse transformation of the first(last) joint.

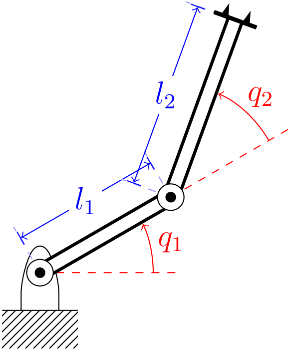
$$R_0^n = R_0^1 R_1^2 \dots R_{n-1}^n = R_g$$

$$(R_0^1)^{-1} R_o^n = (R_0^1)^{-1} R_g \text{ or } R_o^n (R_{n-1}^n)^{-1} = R_g (R_{n-1}^n)^{-1}$$

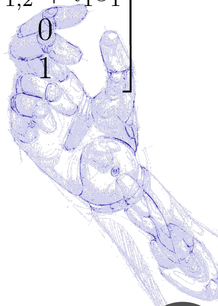
And we try to isolate again

Inverse kinematics model

Examples



$$R(q_1, q_2) = \begin{bmatrix} c_{1,2} & -s_{1,2} & 0 & l_2 c_{1,2} + l_1 c_1 \\ s_{1,2} & c_{1,2} & 0 & l_2 s_{1,2} + l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Inverse kinematics model

Analytical solution

$$x = l_2 c_{1,2} + l_1 c_1$$
$$y = l_2 s_{1,2} + l_1 s_1$$

Inverse kinematics model

Analytical solution

$$x = l_2 c_{1,2} + l_1 c_1$$
$$y = l_2 s_{1,2} + l_1 s_1$$

$$x^2 + y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 (c_1 c_{1,2} + s_1 s_{1,2})$$

Inverse kinematics model

Analytical solution

$$x = l_2 c_{1,2} + l_1 c_1$$
$$y = l_2 s_{1,2} + l_1 s_1$$

$$x^2 + y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 (c_1 c_{1,2} + s_1 s_{1,2})$$

Remember:

$$cos(\alpha - \beta) = cos\alpha cos\beta + sin\alpha sin\beta$$

Inverse kinematics model

Analytical solution

$$x = l_2 c_{1,2} + l_1 c_1$$
$$y = l_2 s_{1,2} + l_1 s_1$$

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Inverse kinematics model

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$$cos(\alpha - \beta) = cos\alpha cos\beta + sin\alpha sin\beta$$

Therefore:

$$x^2 + y^2 = l_1^2 + l_2^2 + 2l_1 l_2 c_2$$

$$c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1 l_2}$$

$$q_2 = cos^{-1} \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1 l_2}$$

Inverse kinematics model

Analytical solution

$$x = l_2 c_{1,2} + l_1 c_1$$
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Remember:

$$cos(\alpha + \beta) = cos\alpha cos\beta - sin\alpha sin\beta$$
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Inverse kinematics model

Analytical solution

$$x = l_2 c_{1,2} + l_1 c_1$$
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Therefore:

$$x = l_2 c_1 c_2 - l_2 s_1 s_2 + l_1 c_1$$
$$y = l_2 c_1 s_2 + l_2 c_2 s_1 + l_1 s_1$$

Inverse kinematics model

Analytical solution

We can write:

$$x = k_1 c_1 - k_2 s_1$$
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Inverse kinematics model

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Where:

$$k_1 = l_1 + l_2 c_2$$
$$k_2 = l_2 s_2$$

We define:

$$r = \sqrt{k_1 + k_2}$$
$$\beta = atan2(k_2, k_1)$$

Inverse kinematics model

Analytical solution

We can write:

$$x = k_1 c_1 - k_2 s_1$$
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Then:

$$k_1 = r cos\beta$$
$$k_2 = r sin\beta$$

Inverse kinematics model

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$$k_1 = r cos\beta$$
$$k_2 = r sin\beta$$

So we can write:

$$\frac{x}{r} = cos\beta c_1 - sin\beta s_1$$
$$\frac{y}{r} = cos\beta s_1 + sin\beta c_1$$

Inverse kinematics model

Analytical solution

We can write:

$$x = k_1 c_1 - k_2 s_1$$
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Where:

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Or:

$$\frac{x}{r} = cos(\beta + q_1)$$
$$\frac{y}{r} = sin(\beta + q_1)$$

Inverse kinematics model

Analytical solution

Are we there yet??

Inverse kinematics model

Analytical solution

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$$\frac{x}{r} = cos(\beta + q_1)$$
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Inverse kinematics model

Analytical solution

Are we there yet??

$$\frac{x}{r} = cos(\beta + q_1)$$
$$\frac{y}{r} = sin(\beta + q_1)$$

$$\beta + q_1 = atan2(\frac{y}{r}, \frac{x}{r}) = atan2(y, x)$$

Inverse kinematics model

Analytical solution

Are we there yet??

$$\frac{x}{r} = cos(\beta + q_1)$$
$$\frac{y}{r} = sin(\beta + q_1)$$

$$\beta + q_1 = atan2(\frac{y}{r}, \frac{x}{r}) = atan2(y, x)$$

$$q_1 = atan2(x, y) - \beta = atan2(y, x) - atan2(k_2, k_1)$$

Inverse kinematics model

Analytical solution

$$q_2 = cos^{-1}\frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1l_2}$$

$$q_1 = atan2(x, y) - \beta = atan2(y, x) - atan2(k_2, k_1)$$

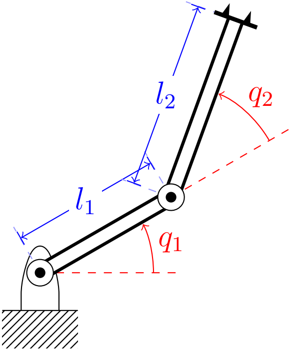
Where:

$$k_1 = l_1 + l_2cos(q_2)$$
$$k_2 = l_2sin(q_2)$$

Inverse kinematics model

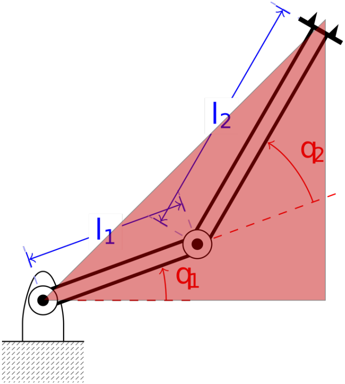
Geometric solutions

We could have solved the previous problem using geometry



Inverse kinematics model

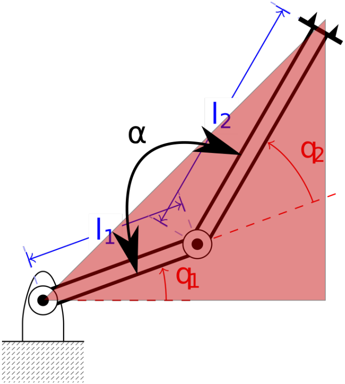
Geometric solutions



We define the length r using Pythagoras:
$r^2 = x^2 + y^2$

Inverse kinematics model

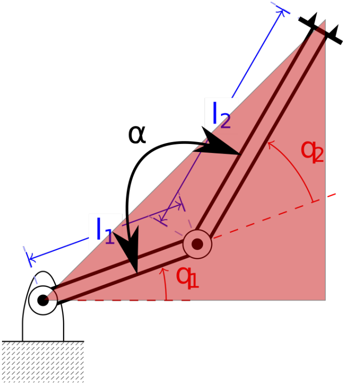
Geometric solutions



We calculate the angle α using the cosine law:

Inverse kinematics model

Geometric solutions

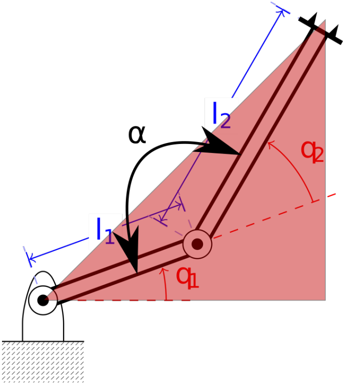


We calculate the angle α using the cosine law:

$$r^2 = l_1^2 + l_2^2 - 2l_1l_2cos\alpha$$

Inverse kinematics model

Geometric solutions



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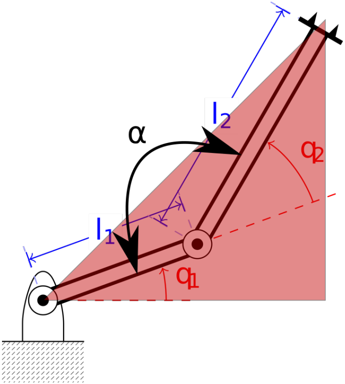
$$r^2 = l_1^2 + l_2^2 - 2 l_1 l_2 cos\alpha$$

$$cos\alpha = \frac{l_1^2 + l_2^2 - r^2}{2 l_1 l_2}$$

Inverse kinematics model

Geometric solutions



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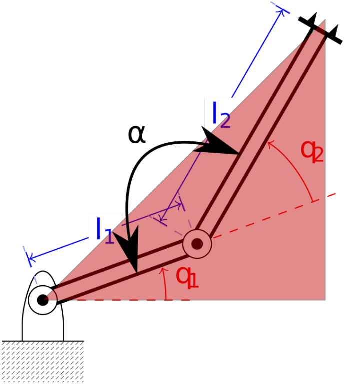
$$r^2 = l_1^2 + l_2^2 - 2l_1l_2cos\alpha$$

$$cos\alpha = \frac{l_1^2 + l_2^2 - r^2}{2l_1l_2}$$

$$cos\alpha = \frac{l_1^2 + l_2^2 - x^2 - y^2}{2l_1l_2}$$

Inverse kinematics model

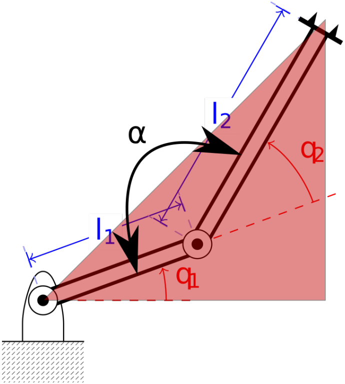
Geometric solutions



We know that: $\alpha = \pi - q_2$

Inverse kinematics model

Geometric solutions



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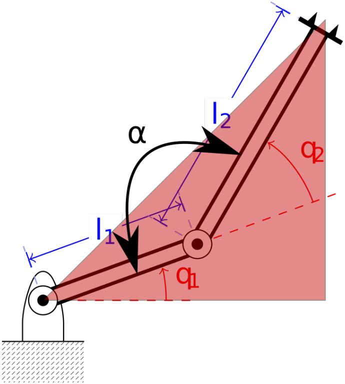
And: $cos(\pi - q) = -cosq$

Inverse kinematics model

Geometric solutions



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And: $cos(\pi - q) = -cosq$

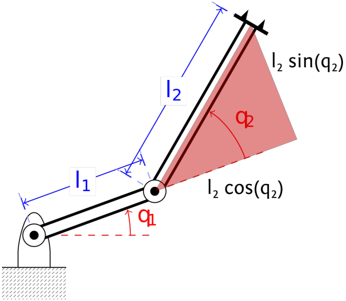
Therefore:

$$cosq_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1l_2}$$

$$q_2 = cos^{-1}\frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1l_2}$$

Inverse kinematics model

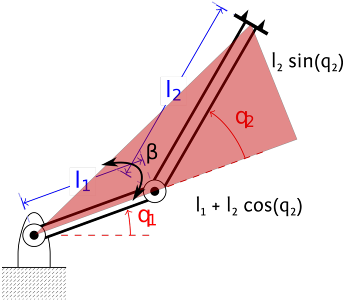
Geometric solutions



We know that the base of this triangle is $l_2 cos q_2$ while the height of the triangle is $l_2 sin q_2$

Inverse kinematics model

Geometric solutions



For this triangle, we can calculate the angle β:

$$\beta = tan^{-1}\frac{l_2 sin q_2}{l_1 + l_2 cos q_2}$$

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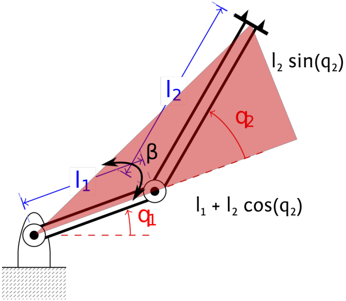

For this triangle, we can calculate the angle β:

$$\beta = tan^{-1}\frac{l_2 sinq_2}{l_1 + l_2 cosq_2}$$

From our initial triangle we know that:

$$tan^{-1}\frac{y}{x} = \beta + q_1$$

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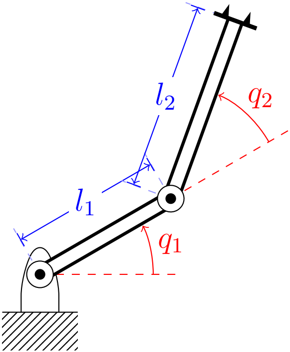

We therefore have:

$$q1 = tan^{-1}\frac{y}{x} - tan^{-1}\frac{l_2 sin q_2}{l_1 + l_2 cos q_2}$$

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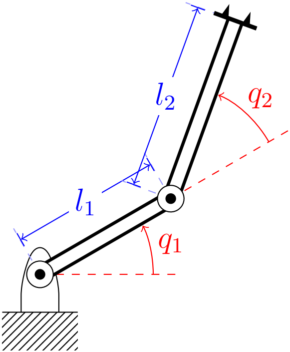

We therefore have:

$$q1 = tan^{-1}\frac{y}{x} - tan^{-1}\frac{l_2 sinq_2}{l_1 + l_2 cosq_2}$$

And we already know the relationship of q_2 in terms of the position:

$$q_2 = cos^{-1}\frac{l_1^2 + l_2^2 - x^2 - y^2}{2l_1l_2}$$

Questions?