Finite Element Analysis

An introduction

UNIVERSITATEA BABEȘ-BOLYAI

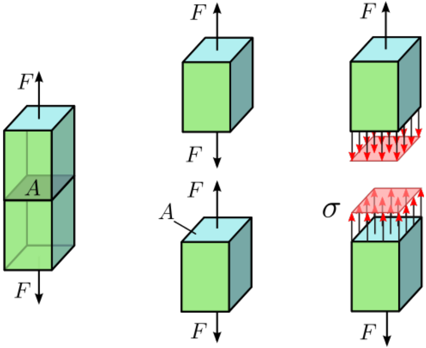
Last update: March 14, 2023

Agenda

- Strength of materials
- Analytical solutions
- Non-standard shapes
- Finite element principles
- Formulations

Strength of materials

Material properties



Strength of materials

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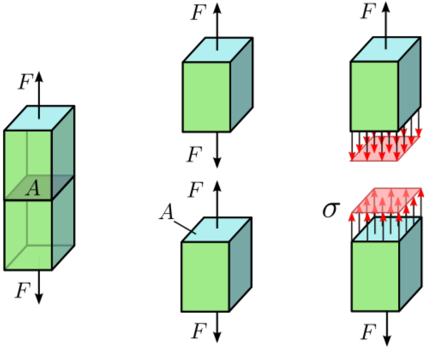

If we know Young's modulus (E), we can calculate strain (ϵ)as a function of stress (σ)

Strength of materials

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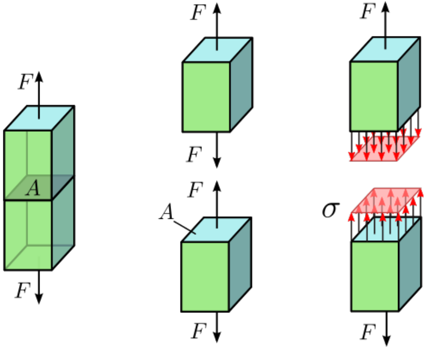

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Strength of materials

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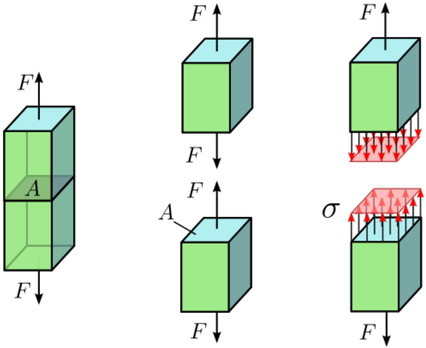

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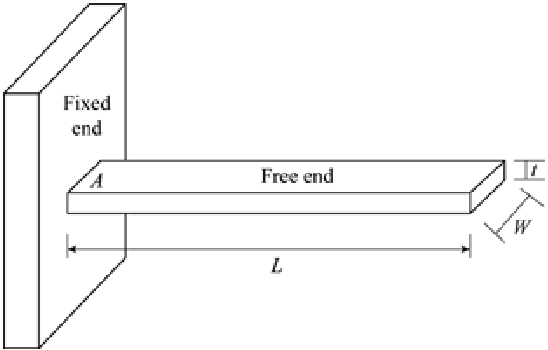
$\sigma = E\epsilon$

$$\frac{F}{A} = E\epsilon$$

Strength of materials

Cantilever beam



Strength of materials

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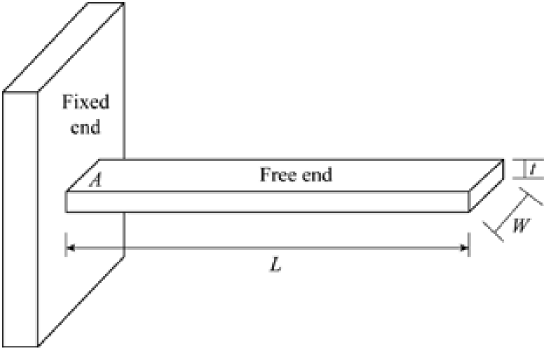
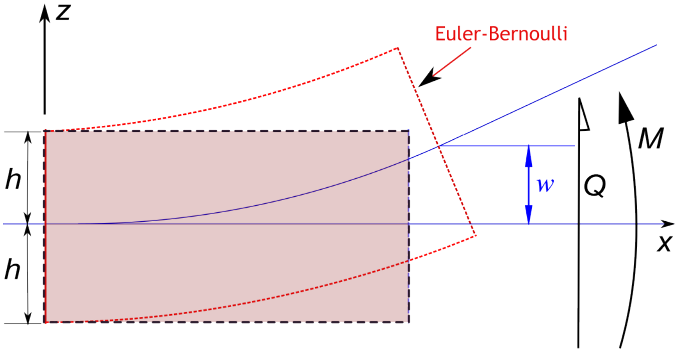


How do we calculate the deformation of a cantilever beam under known load?

Strength of materials

Timoshenko-Ehrenfest beam theory



Strength of materials

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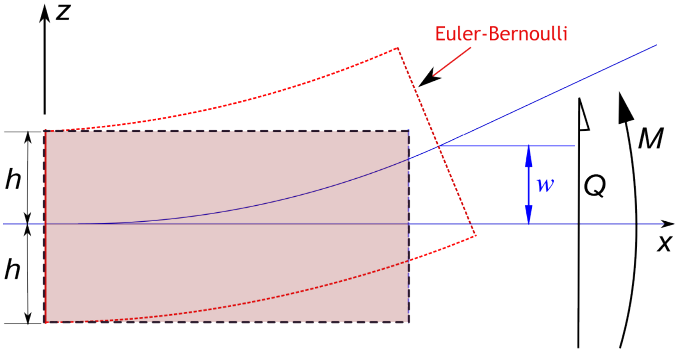

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right) = q(x)$$

Why is Euler-Bernouli wrong?

Strength of materials

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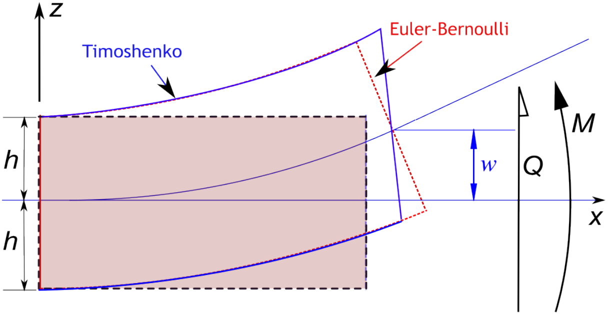

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$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right) = q(x)$$
$$\frac{\mathrm{d}w}{\mathrm{d}x} = \varphi - \frac{1}{\kappa AG}\frac{\mathrm{d}}{\mathrm{d}x}\left(EI\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right).$$

Strength of materials

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- A is the cross section area.
- E is the elastic modulus.
- G is the shear modulus.
- I is the second moment of area.
- κ, called the Timoshenko shear coefficient, depends on the geometry. Normally, $\kappa = 5/6$ for a rectangular section.
- $q(x)$ is a distributed load (force per length).

Strength of materials

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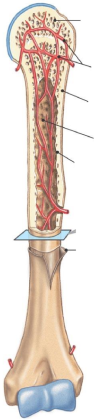
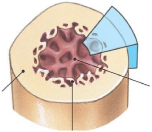
Analytical solutions possible

Strength of materials

Back to biomechanics





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Can we apply this to biological materials?

Strength of materials

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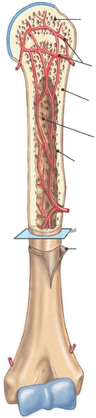

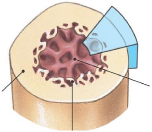

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Can we apply this to biological materials?

Why not?

Finite Element Analysis

Description

A computational scheme to solve field problems. The field can be *stress*, *heat*, *pressure*, *electric*, *magnetic*, etc, etc. The principle involves dividing the body in finite pieces that can provide analytical solutions.

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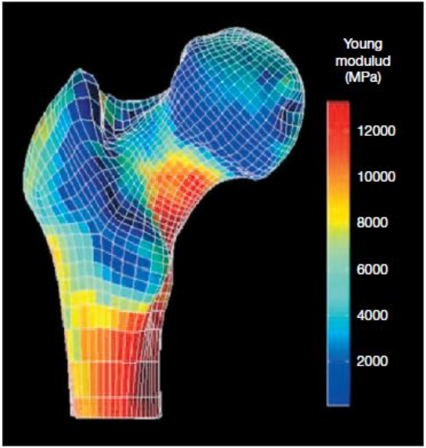
Key word is **discretization**

Finite Element Analysis

From cantilever beams to bones



Finite Element Analysis

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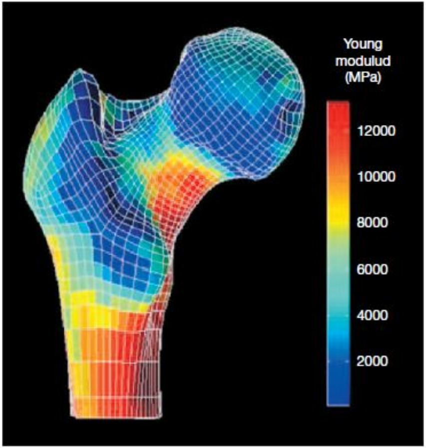

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Finite Element Analysis

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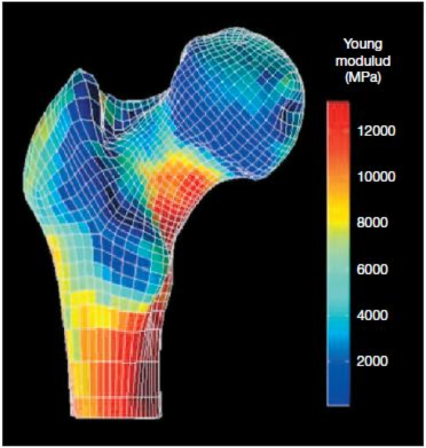

We approximate a complex geometry with a combination of simple ones

What are the limitations of this approach?

Finite Element Analysis

Different levels of discretization

Finite Element Analysis

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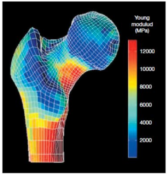
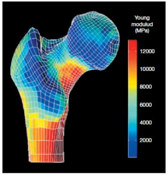


- Geometry

Finite Element Analysis

Different levels of discretization



- Geometry
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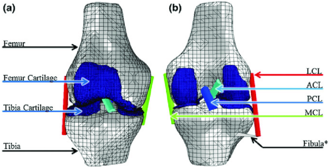
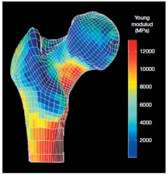
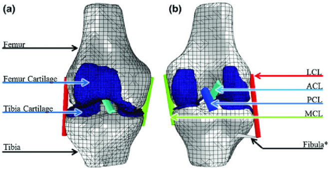


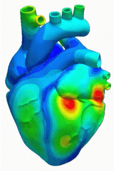
Finite Element Analysis

Different levels of discretization

- Geometry
- Materials
- Time







Finite Element Analysis

Geometry discretization

To discretize geometry, we have several elements available (Think of them as lego blocks):

Finite Element Analysis

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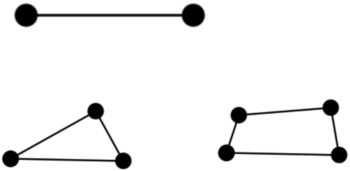
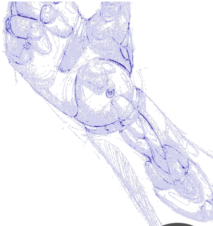


- 1D (Rods, beams, Trusses, Frames)
- 2D (Triangular, Quadrilateral, Plates, Shells)



Finite Element Analysis

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To discretize geometry, we have several elements available (Think of them as lego blocks):

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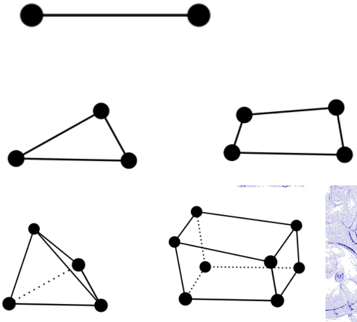
- 3D (Tetrahedral, Hexahedral)



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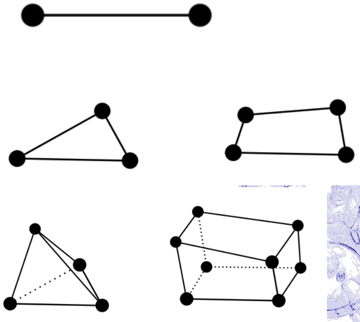

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What does it mean 1D, 2D, 3D?

A model of a spring.

Finite Element Analysis

1D element equations

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We need to write a force displacement equation for each 'node'

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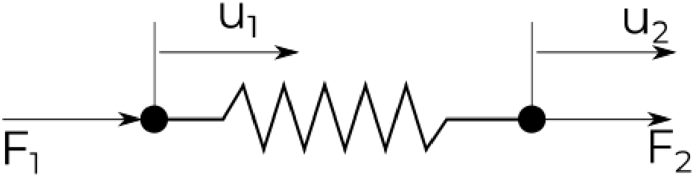

Finite Element Analysis

1D element equations

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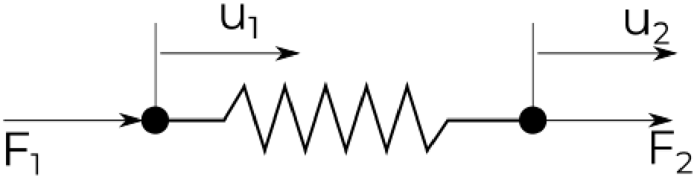

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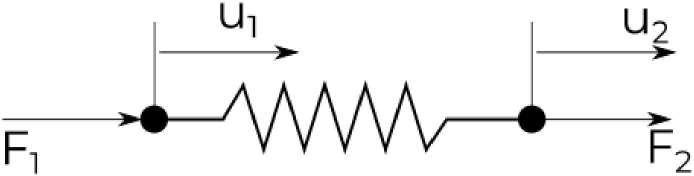

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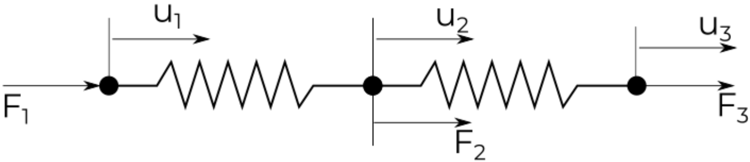
$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Finite Element Analysis

Combining elements



How do we write the equations for two springs?

Finite Element Analysis

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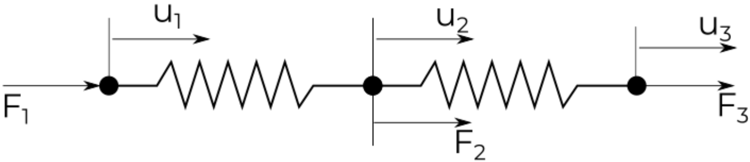

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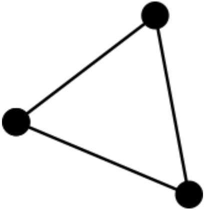
$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Finite Element Analysis

2D element equations



Finite Element Analysis

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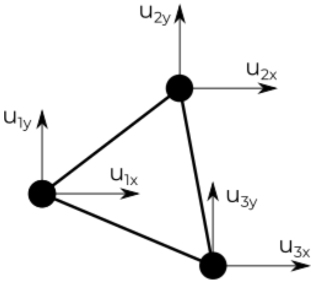

Finite Element Analysis

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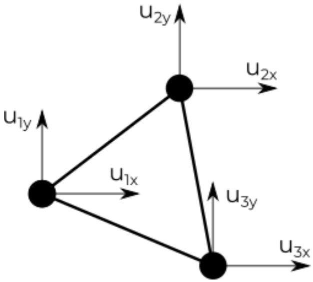
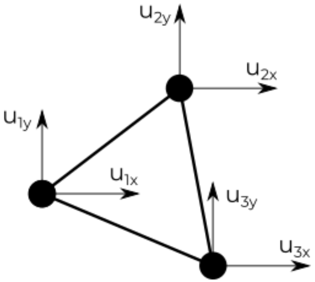


We assume linear transition of the stress/strain between the nodes of the element

Finite Element Analysis

2D element equations



$$k = tA(B^T EB)$$

where:

- t: Thickness of the plate
- A: Area of the triangle
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Finite Element Analysis

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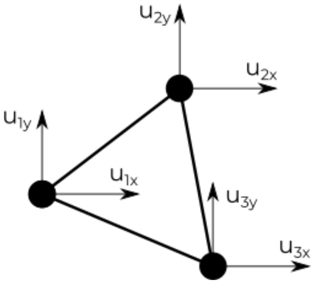

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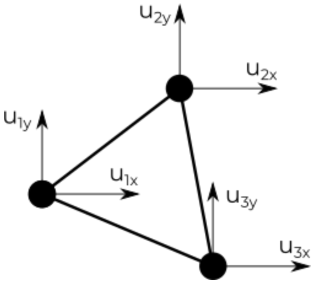
$$B = \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{31} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{13} \end{bmatrix}$$

Finite Element Analysis

2D element equations



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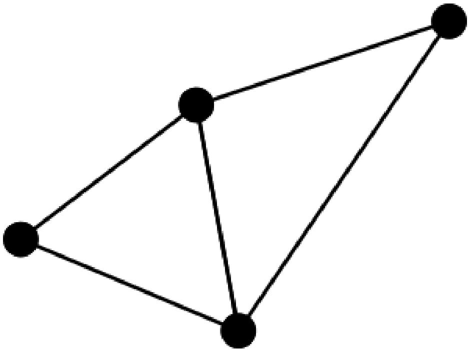
$x_{ij} = x_i - x_j$

Finite Element Analysis

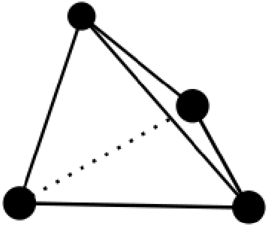
Combining elements



How do we combine elements?

Finite Element Analysis

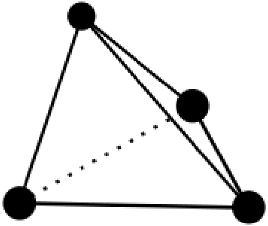
3D element equations



We follow a similar logic for higher degree elements.

Finite Element Analysis

3D element equations



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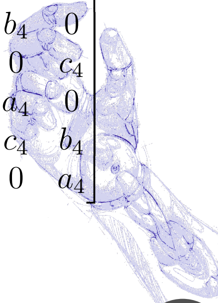
$$B = \frac{1}{6V}\begin{bmatrix} a_1 & 0 & 0 & a_2 & 0 & 0 & a_3 & 0 & 0 & a_4 & 0 & 0 \\ 0 & b_1 & 0 & 0 & b_2 & 0 & 0 & b_3 & 0 & 0 & b_4 & 0 \\ 0 & 0 & c_1 & 0 & 0 & c_2 & 0 & 0 & c_3 & 0 & 0 & c_4 \\ b_1 & a_1 & 0 & b_2 & a_2 & 0 & b_3 & a_3 & 0 & b_4 & a_4 & 0 \\ 0 & c_1 & b_1 & 0 & c_2 & b_2 & 0 & c_3 & b_3 & 0 & c_4 & b_4 \\ c_1 & 0 & a_1 & c_2 & 0 & a_2 & c_3 & 0 & a_3 & c_4 & 0 & a_4 \end{bmatrix}$$



Finite Element Analysis

3D element equations

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$a_1 = y_2 z_{43} - y_3 z_{42} + y_4 z_{32}, b_1 = -x_2 z_{43} + x_3 z_{42} - x_4 z32, c_1 =$
$...a_2 = ..., b_2 = ..., c_2 = ...a_3 = ..., b_3 = ..., c_3 = ...$

Finite Element Analysis

Problem construction

In any of these cases, we are trying to solve a problem of force and displacement

$$\left[K\right]\left\{u\right\} = \left\{F\right\}$$

Finite Element Analysis

Material discretization

How do we model different materials?

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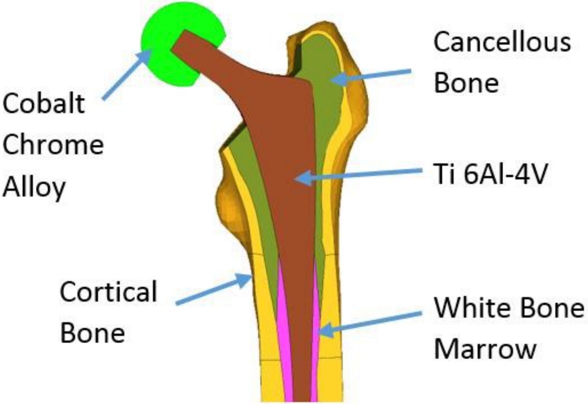

Finite Element Analysis

Time discretization

What about dynamic simulations?

Finite Element Analysis

Time discretization

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$$\left[K\right]\left\{u\right\}=\left\{F\right\}$$

Finite Element Analysis

Time discretization

What about dynamic simulations?

$$\left[K\right]\left\{u\right\} = \left\{F\right\}$$

How can we capture dynamics?

Finite Element Analysis

Time discretization

What about dynamic simulations?

$$\left[K\right]\left\{u\right\} = \left\{F\right\}$$

How can we capture dynamics?

$$\left[M\right]\left\{\ddot{u}\right\} + \left[K\right]\left\{u\right\} = \left\{F\right\}$$

What about dynamic simulations?

$$[K]\{u\} = \{F\}$$

How can we capture dynamics?

$$[M]\{\ddot{u}\} + [K]\{u\} = \{F\}$$

And if we have viscous effects?

Finite Element Analysis

Time discretization

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Finite Element Analysis

Time discretization

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How do we construct matrices M and C?

Finite Element Analysis

Applications in biomechanics

Why is this useful for biomechanics?

Finite Element Analysis

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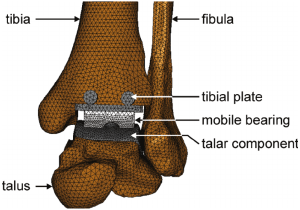

Finite Element Analysis

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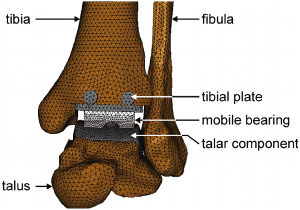

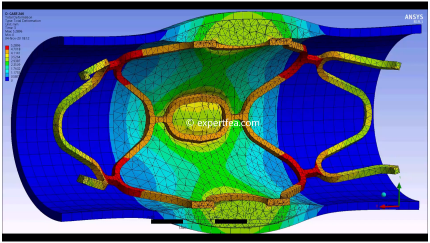

Finite Element Analysis

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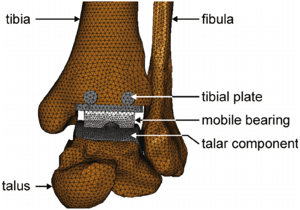

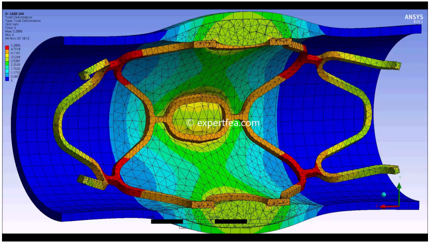

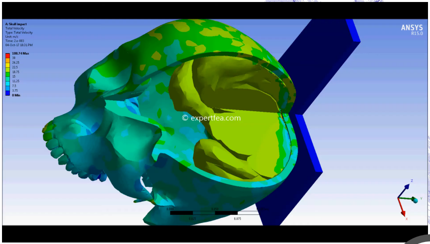

Finite Element Analysis

Pitfalls

What are some disadvantages of FEA?

Coming up next

Material properties of soft and hard tissues

Questions?