



Robot dynamic modeling



Last update: November 23, 2023

Agenda

- Lagrangian of a robot
- How it all fits together



What did we do last week?

Recap

Lagrange defined a basic quantity for any system of bodies as the difference between its kinetic and potential energy.

$$L = K - P$$

We call this quantity the Lagrangian of the system.



What did we do last week?

Recap

We define the Lagrangian as the difference between Kinetic and Potential energy of our system

$$L = K - P$$

where:

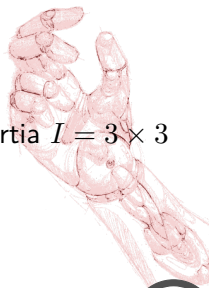
Potential Energy

$$P = mgh$$

Kinetic Energy

$$K = \frac{1}{2}(mu^2 + \omega^T I \omega)$$

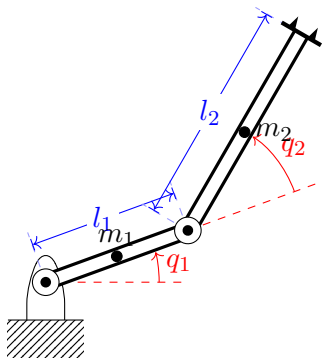
Moment of inertia $I = 3 \times 3$



Lagrangian of a robot

How do we calculate it?

Let's take an 'easy' example of a 2-link planar robot.



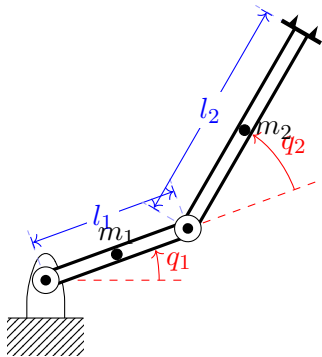
Let's assume that segments have masses m_1 and m_2 respectively.



Lagrangian of a robot

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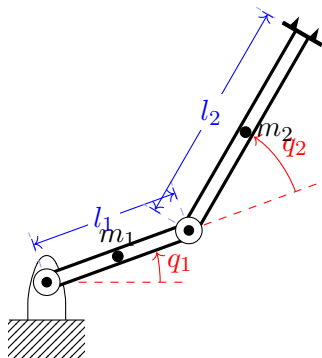


We need to calculate its Lagrangian

Lagrangian of a robot

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Let's assume that segments have masses m_1 and m_2 respectively.

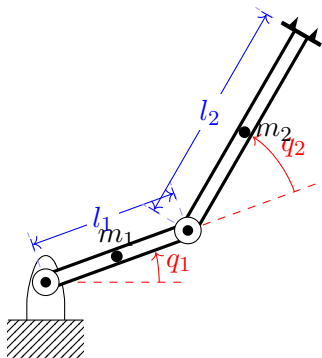


We need to calculate its Lagrangian in terms of some 'generalized' coordinates.

Lagrangian of a robot

How do we calculate it?

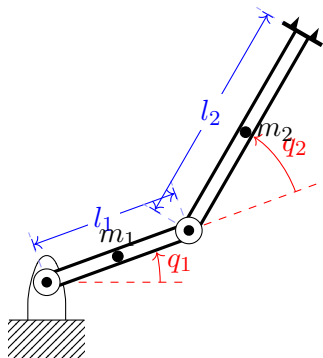
Which 'generalized' coordinates are most convenient?



Lagrangian of a robot

How do we calculate it?

Which 'generalized' coordinates are most convenient?



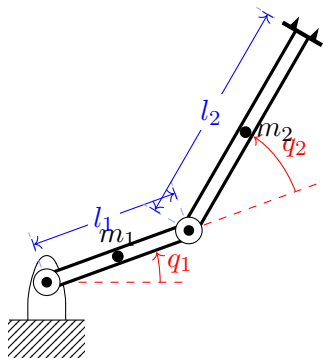
X, Y ?



Lagrangian of a robot

How do we calculate it?

Which 'generalized' coordinates are most convenient?



$X, Y?$

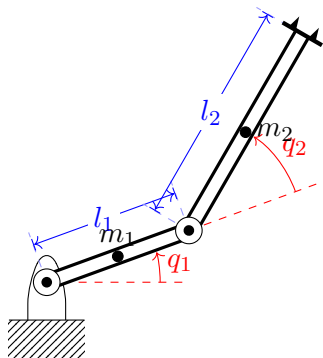
$\theta, r?$



Lagrangian of a robot

How do we calculate it?

Which 'generalized' coordinates are most convenient?



$X, Y?$

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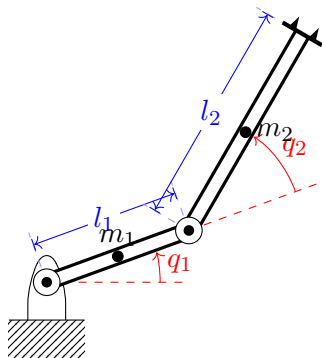
$q_1, q_2?$



Lagrangian of a robot

How do we calculate it?

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$X, Y?$

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$q_1, q_2?$

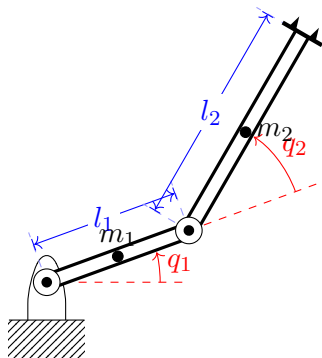
Why?



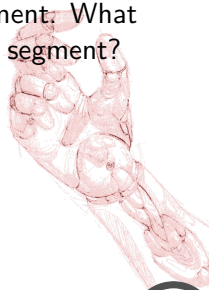
Lagrangian of a robot

Potential energy

We need to calculate the total potential energy of the system with respect to q, \dot{q} .

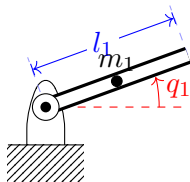


The total potential energy is the sum of the potential energies of each segment. What is the potential energy of each segment?



Lagrangian of a robot

Potential energy

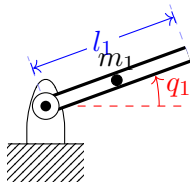


We consider the mass of the link to be concentrated at its center of mass.



Lagrangian of a robot

Potential energy



We consider the mass of the link to be concentrated at its center of mass.

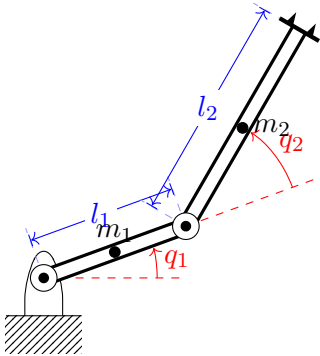
Therefore:

$$P_1(q, \dot{q}) = m_1 g \frac{l_1}{2} \sin q_1$$



Lagrangian of a robot

Potential energy

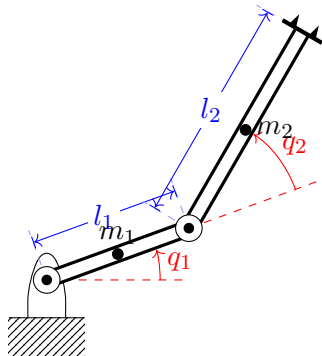


For the second segment, we think alike:



Lagrangian of a robot

Potential energy



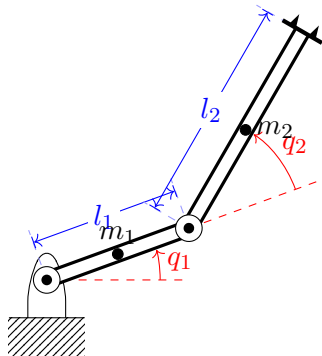
For the second segment, we think alike:

$$P_2(q, \dot{q}) = m_2 g \left(l_1 \sin q_1 + \frac{l_2}{2} \sin(q_1 + q_2) \right)$$



Lagrangian of a robot

Potential energy



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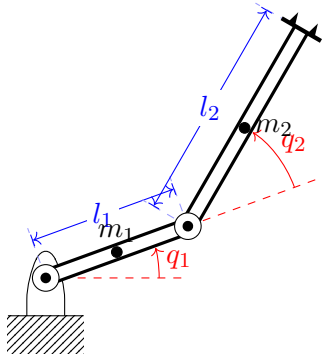
The total potential energy is therefore:

$$P(q, \dot{q}) = m_1 g \frac{l_1}{2} \sin q_1 + m_2 g \left(l_1 \sin q_1 + \frac{l_2}{2} \sin(q_1 + q_2) \right)$$



Lagrangian of a robot

Potential energy



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Does P depend on \dot{q} ?

Lagrangian of a robot

Kinetic energy

Once again, we take the kinetic energy of each segment with respect to q, \dot{q} and add them together.

$$K_{total}(q, \dot{q}) = K_1(q, \dot{q}) + K_2(q, \dot{q})$$



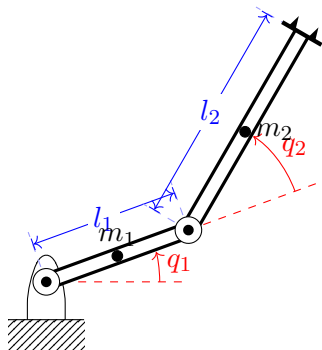
Lagrangian of a robot

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Let's start with the linear kinetic energy first.



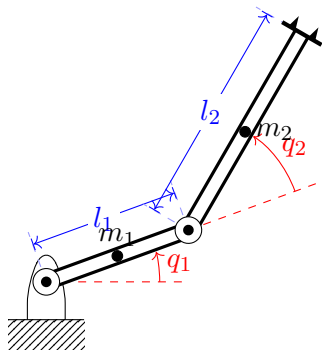
Lagrangian of a robot

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$$K_{lin}(q, \dot{q}) = \frac{1}{2} \sum_{i=1}^n [u_i^T m_i u_i]$$



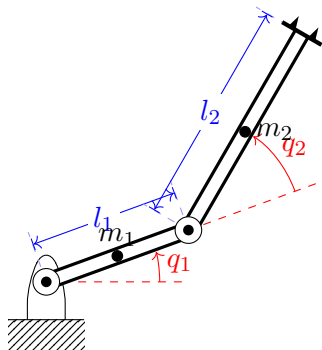
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Do we know what is u in terms of q, \dot{q} ?



Lagrangian of a robot

Going back in time

How do we convert the linear end-effector velocity (u) into joint velocity (\dot{q})?



Lagrangian of a robot

Going back in time

How do we convert the linear end-effector velocity (u) into joint velocity (\dot{q})?

The jacobian!

$$u = J_u \dot{q}$$



Lagrangian of a robot

Going back in time

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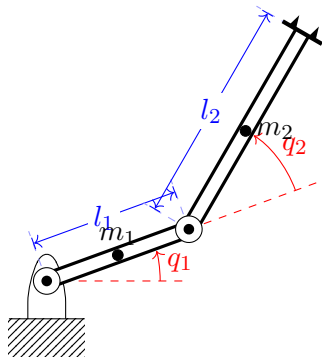
$$u = J_u \dot{q}$$

But what Jacobian? What velocities do we want to calculate?



Lagrangian of a robot

Kinetic energy

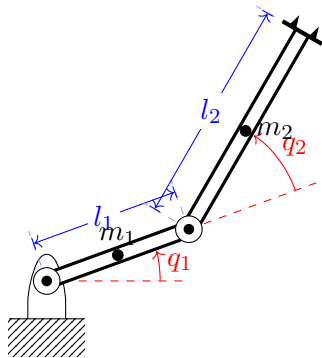


Since we need the velocity of the masses, we calculate the Jacobian *until* the masses.



Lagrangian of a robot

Kinetic energy



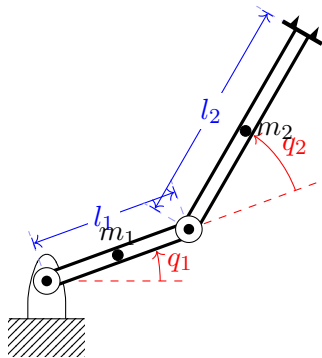
Since we need the velocity of the masses, we calculate the Jacobian *until* the masses.

$$J_{u1} = \left[z_1 \times (o_{m1} - o_1) \right]$$



Lagrangian of a robot

Kinetic energy



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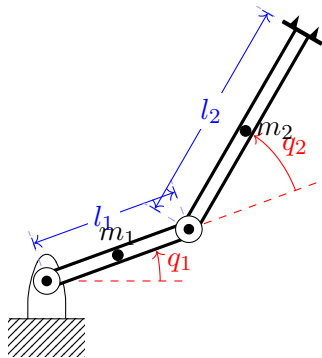
$$J_{u1} = \left[z_1 \times (o_{m1} - o_1) \right]$$

$$J_{u2} = \left[z_1 \times (o_{m2} - o_1) \quad z_2 \times (o_{m2} - o_2) \right]$$



Lagrangian of a robot

Kinetic energy



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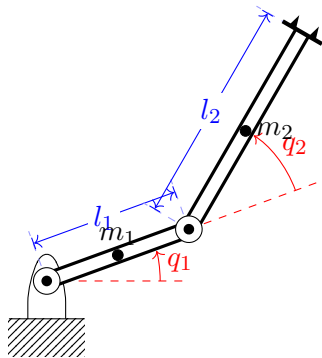
$$J_{u2} = \left[z_1 \times (o_{m2} - o_1) \quad z_2 \times (o_{m2} - o_2) \right]$$

To ensure equal dimensions for the Jacobians, we *pad* the first Jacobian with zeros



Lagrangian of a robot

Kinetic energy



Since we need the velocity of the masses, we calculate the Jacobian *until* the masses.

$$J_{u1} = \begin{bmatrix} z_1 \times (o_{m1} - o_1) \end{bmatrix}$$

$$J_{u2} = \begin{bmatrix} z_1 \times (o_{m2} - o_1) & z_2 \times (o_{m2} - o_2) \end{bmatrix}$$

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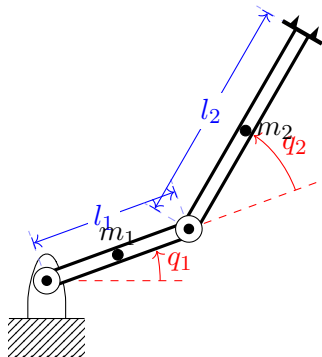
Lagrangian of a robot

Kinetic energy

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$$K_{total}(q, \dot{q}) = K_1(q, \dot{q}) + K_2(q, \dot{q})$$

Let's start with the linear kinetic energy first.



$$K_{lin}(q, \dot{q}) = \frac{1}{2} m u^2 = \frac{1}{2} \dot{q}^T \sum_{i=1}^n [J_{ui}^T m_i J_{ui}] \dot{q}$$



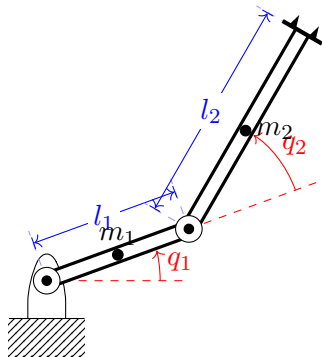
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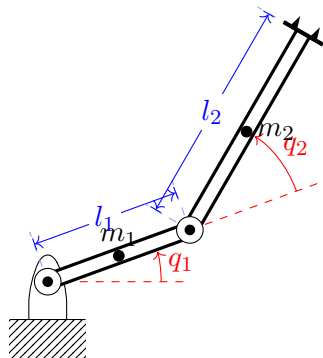
Remember $(AB)^T = B^T A^T$



Lagrangian of a robot

Kinetic energy

... and then the angular kinetic energy:



$$K_{ang}(q, \dot{q}) = \frac{1}{2} \sum_{i=1}^n [\omega_i^T I_i \omega_i]$$



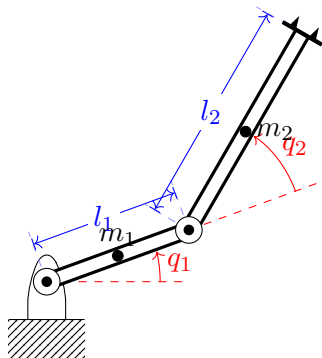
Lagrangian of a robot

Kinetic energy

... and then the angular kinetic energy:

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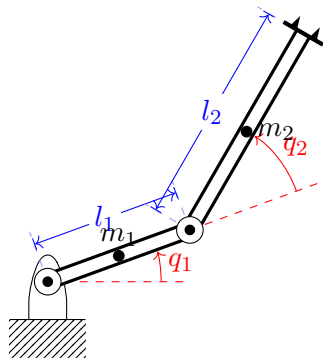
I_i is expressed on the coordinate frame of link i



Lagrangian of a robot

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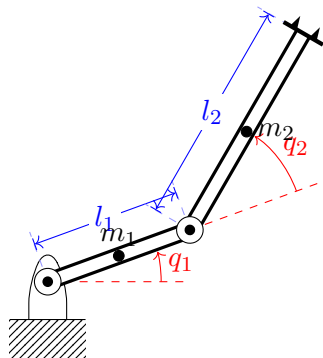
I_i is expressed on the coordinate frame of link i
But must be 'transformed' in the base coordinate frame



Lagrangian of a robot

Kinetic energy

... and then the angular kinetic energy:



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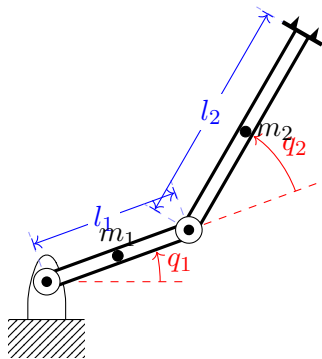
$$I_i^o = R_i I_i^i R_i^T$$



Lagrangian of a robot

Kinetic energy

... and then the angular kinetic energy:



$$K_{ang}(q, \dot{q}) = \frac{1}{2} \sum_{i=1}^n [\omega_i^T I_i \omega_i]$$

I_i is expressed on the coordinate frame of link i
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$$I_i^o = R_i I_i^i R_i^T$$

Therefore

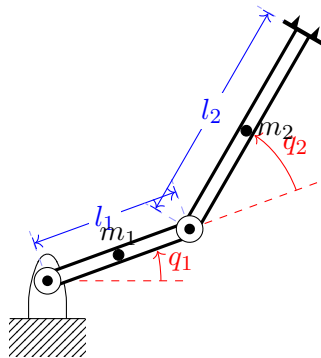
$$K_{ang}(q, \dot{q}) = \frac{1}{2} \dot{q}^T \sum_{i=1}^n [J_{\omega i}^T R_i I_i^i R_i^T J_{\omega i}] \dot{q}$$



Lagrangian of a robot

Kinetic energy

Therefore, the total Kinetic energy of the robot is:



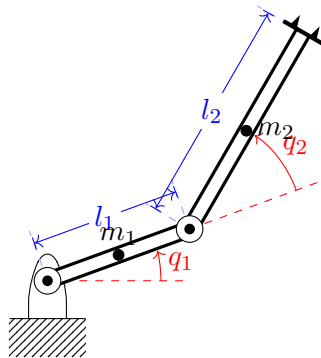
$$K(q, \dot{q}) = \frac{1}{2} \dot{q}^T \sum_{i=1}^n \left[J_{vi}^T m_i J_{vi} + J_{\omega i}^T R_i I_i R_i^T J_{\omega i} \right] \dot{q}$$



Lagrangian of a robot

Kinetic energy

Therefore, the total Kinetic energy of the robot is:



$$K(q, \dot{q}) = \frac{1}{2} \dot{q}^T \sum_{i=1}^n \left[J_{vi}^T m_i J_{vi} + J_{\omega i}^T R_i I_i R_i^T J_{\omega i} \right] \dot{q}$$

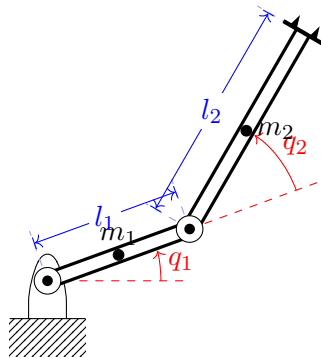
Does K depend on \dot{q} ?



Lagrangian of a robot

Kinetic energy

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Does K depend on \dot{q} ? Does it depend on q ?



Lagrangian of a robot

Jacobians

$$K(q, \dot{q}) = \frac{1}{2} \dot{q}^T \sum_{i=1}^n \left[J_{vi}^T m_i J_{vi} + J_{\omega i}^T R_i I_i R_i^T J_{\omega i} \right] \dot{q}$$

What is the size of J_{vi} and $J_{\omega i}$?



Lagrangian of a robot

Jacobians

$$K(q, \dot{q}) = \frac{1}{2} \dot{q}^T \sum_{i=1}^n \left[J_{vi}^T m_i J_{vi} + J_{\omega i}^T R_i I_i R_i^T J_{\omega i} \right] \dot{q}$$

What is the size of J_{ui} and $J_{\omega i}$?

$$J_{u1} = \begin{bmatrix} z_1 \times (o_3 - o_1) & 0 \end{bmatrix}$$

$$J_{u2} = \begin{bmatrix} z_1 \times (o_3 - o_1) & z_2 \times (o_3 - o_2) \end{bmatrix}$$



Lagrangian of a robot

Jacobians

$$K(q, \dot{q}) = \frac{1}{2} \dot{q}^T \sum_{i=1}^n \left[J_{vi}^T m_i J_{vi} + J_{\omega i}^T R_i I_i R_i^T J_{\omega i} \right] \dot{q}$$

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$$J_{\omega 1} = \begin{bmatrix} z_1 & 0 \end{bmatrix}$$

$$J_{\omega 2} = \begin{bmatrix} z_1 & z_2 \end{bmatrix}$$



Lagrangian of a robot

Let's plug it in the Lagrangian equation of motion

Eventually, we can write the kinetic energy in a condensed format:

$$K(q, \dot{q}) = \frac{1}{2} \dot{q}^T D(q) \dot{q}$$

And the potential energy:

$$P(q) = \sum_{i=1}^n g h_i(q) m_i$$

Therefore, the total Lagrangian is:

$$L(q, \dot{q}) = K - P = \frac{1}{2} \dot{q}^T D(q) \dot{q} - g \sum_{i=1}^n h_i(q) m_i$$



Lagrangian of a robot

Let's plug it all together

If we expand the first term, we get:

$$L = K - P = \frac{1}{2} \dot{q}^T D(q) \dot{q} - g \sum_{i=1}^n h_i(q) m_i$$

$$L = \frac{1}{2} \sum_{i,j} d_{ij}(q) \dot{q}_i \dot{q}_j - g \sum_{i=1}^n h_i(q) m_i$$



Lagrangian of a robot

Let's plug it all together

The equation of motion is:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau$$



Lagrangian of a robot

Let's plug it all together

The equation of motion is:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \tau_k$$



Lagrangian of a robot

Let's plug it all together

The equation of motion is:

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$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \tau_k$$

$$\frac{\partial L}{\partial \dot{q}_k} = \sum_j d_{kj} \dot{q}_j$$



Lagrangian of a robot

Let's plug it all together

The equation of motion is:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \tau_k$$

$$\frac{\partial L}{\partial \dot{q}_k} = \sum_j d_{kj} \dot{q}_j$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = \sum_j d_{kj} \ddot{q}_j + \sum_j \frac{d}{dt} d_{kj} \dot{q}_j = \sum_j d_{kj} \ddot{q}_j + \sum_{i,j} \frac{\partial d_{kj}}{\partial q_i} \dot{q}_i \dot{q}_j$$



Lagrangian of a robot

Let's plug it all together

The second term is:

$$\frac{\partial L}{\partial q_k} = \frac{1}{2} \sum_{i,j} \frac{\partial d_{ij}}{\partial q_k} \dot{q}_i \dot{q}_j - \frac{\partial P}{\partial q_k}$$

Therefore, everything together is:

$$\sum_j d_{kj} \ddot{q}_j + \sum_{i,j} \left\{ \frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2} \frac{\partial d_{ij}}{\partial q_k} \right\} \dot{q}_i \dot{q}_j + \frac{\partial P}{\partial q_k} = \tau_k$$



Lagrangian of a robot

Condensed form

We can write this equation in a more general form:

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau$$



Lagrangian of a robot

Condensed form

We can write this equation in a more general form:

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau$$

The matrix D , contains information about the **inertia** of the system, therefore contains all the masses and moments of inertia.



Lagrangian of a robot

Condensed form

We can write this equation in a more general form:

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau$$

The matrix D , contains information about the **inertia** of the system, therefore contains all the masses and moments of inertia.

The matrix C has elements related to the **centrifugal** and **Coriolis** terms



Lagrangian of a robot

Condensed form

We can write this equation in a more general form:

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau$$

The matrix D , contains information about the **inertia** of the system, therefore contains all the masses and moments of inertia.

The matrix C has elements related to the **centrifugal** and **Coriolis** terms

Finally, the term g contains the dependence of the potential energy from the position of the robot.



Lagrangian of a robot

Christoffel symbols

The k, j -th element of matrix $C(q, \dot{q})$ is defined as:

$$c_{kj} = \sum_{i=1}^n c_{ijk}(q) \dot{q}_i$$

$$= \sum_{i=1}^n \frac{1}{2} \left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right\} \dot{q}_i$$



Inverse and forward dynamics

Again the inverse?



Inverse and forward dynamics

Again the inverse?

Which one is which?



Inverse and forward dynamics

Again the inverse?

Which one is which?

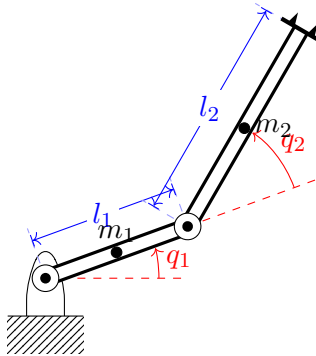
$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau$$

$$D(q)^{-1}(\tau - C(q, \dot{q})\dot{q} - g(q)) = \ddot{q}$$



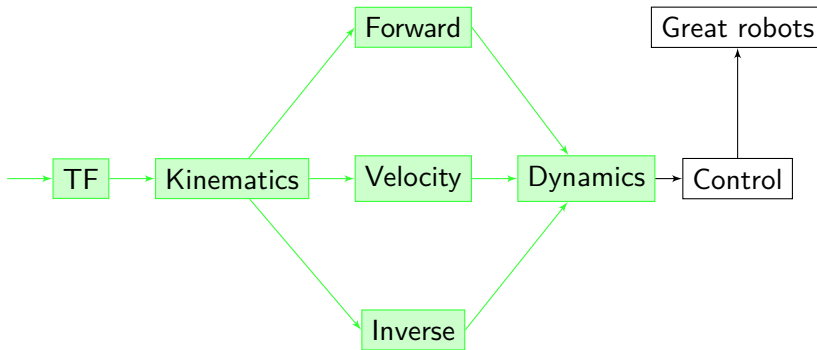
Lagrangian of a robot

Let's apply it on this robot



Grand scheme

The big picture





Questions?