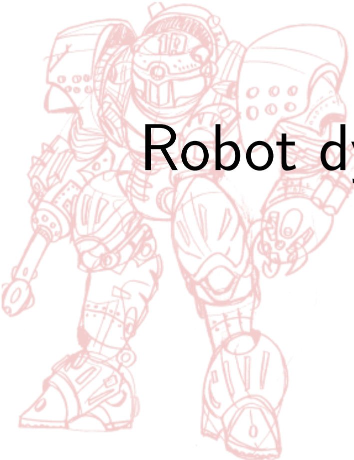

Robot dynamic modeling

TECHNICAL UNIVERSITY
OF CLUJ-NAPOCA
ROMANIA

Last update: November 16, 2023

Agenda

- Background
- Dynamics
- The Lagrangian
- Dynamic and Kinetic energy
- Inertia and moments of inertia

What did we do so far?

Recap

We defined a matrix that we called the DGM, which helps us calculate the pose of the end-effector of a robot as a function of the joint coordinates

$$R_0^n = \begin{bmatrix} & rotation & & \begin{matrix} trans- \\ la- \\ tion \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What did we do so far?

Recap

We defined a matrix that we called the Jacobian, that maps joint velocities to end-effector velocities.

$$\xi = J\dot{q}$$

What did we do so far?

Recap

We found a way to calculate the inverse kinematics and velocities models.

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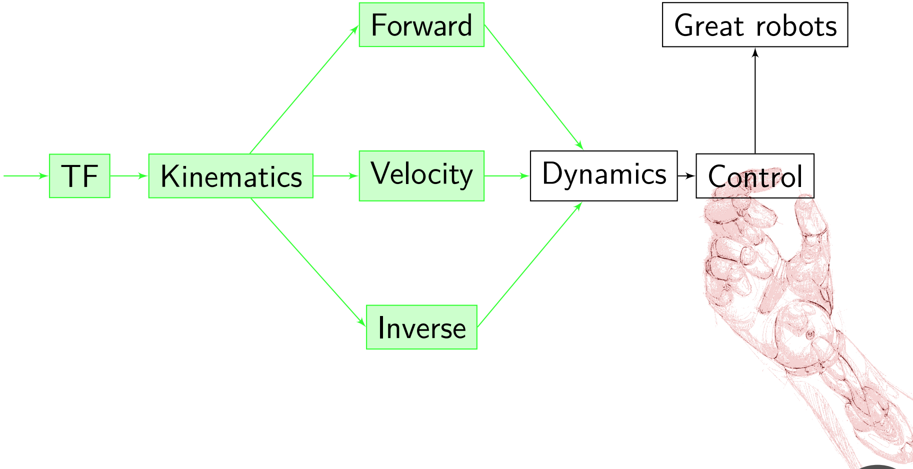
Recap

We found a way to calculate the inverse kinematics and velocities models.

We saw how to use all this for calculating trajectories for our robots.

Grand scheme

The big picture



Dynamic modeling

What is it all about?

Kinematics:

Dynamics (Kinetics):

Dynamic modeling

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Kinematics: description of motion of bodies or system of bodies

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Dynamic modeling

What is it all about?

Kinematics: description of motion of bodies or system of bodies (Forward/Inverse **kinematics**, velocities, trajectories)

Dynamics (Kinetics): description of the causes resulting in those motions (i.e. forces and torques)

Dynamic modeling

What is it all about?

Dynamic model

A system of differential equations that gives us the relationship between input joint forces/torques and resulting joint accelerations.

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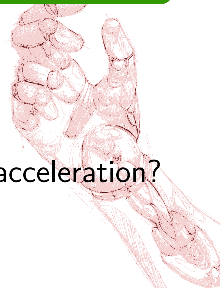


Why is this useful?

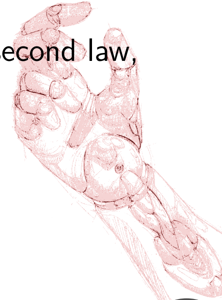
Do you know of any equation that relates force with acceleration?

Dynamic modeling

Newton's equations

$$\sum F = m\ddot{x}$$

This is the famous equation derived from Newton's second law, which relates force and acceleration.



Dynamic modeling

Newton's equations

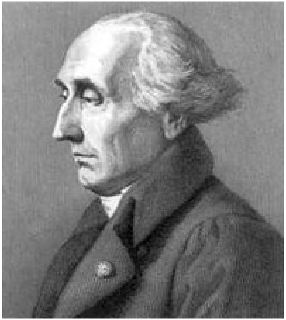
$$\sum F = m\ddot{x}$$

This is the famous equation derived from Newton's second law, which relates force and acceleration.

We won't be using it though due to the complex equations it will result when modeling kinematic chains.

Dynamic modeling

Lagrange-Euler formulation of mechanics



Between 1772 and 1788, Lagrange formulated mechanics in a more general way, more suitable for robotics later on.

Lagrangian mechanics

A more sophisticated formulation of mechanics

Lagrange defined a basic quantity for any system of bodies as the difference between its kinetic and potential energy.

$$L(q, \dot{q}) = K(q, \dot{q}) - P(q)$$

We call this quantity the Lagrangian of the system.

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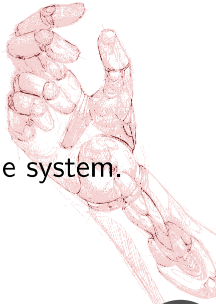
$$L(q, \dot{q}) = K(q, \dot{q}) - P(q)$$

We call this quantity the Lagrangian of the system.

q and $\dot{q}$ are the *generalized coordinates* or *state* of the system.



Lagrangian mechanics

A more sophisticated formulation of mechanics

Using this quantity, we can describe the evolution of any system of bodies under the influence of a set of external forces using the following equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau$$

Definitions

Potential energy

Potential energy

The energy possessed by an object because of its position relative to other objects, stresses within itself, its electric charge, or other factors.

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The energy possessed by an object because of its position relative to other objects, stresses within itself, its electric charge, or other factors.

In robotics, we deal with rigid objects, electrically neutral. What are the sources of potential energy?

Definitions

Potential energy

The most common source of potential energy in robotics is the gravitational field of Earth.

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Where m is the mass of the object, g is the gravitational constant, and h is the height of the object.

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But height from where? What is the reference?

Definitions

Potential energy

Reference for potential

Potential is only important when considering the difference of potential. Therefore, the reference is not important, as long as it does not change over time, and we use the same one for all the objects.

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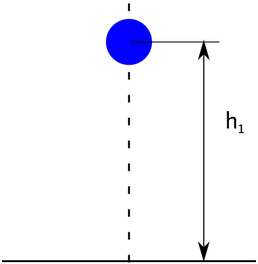

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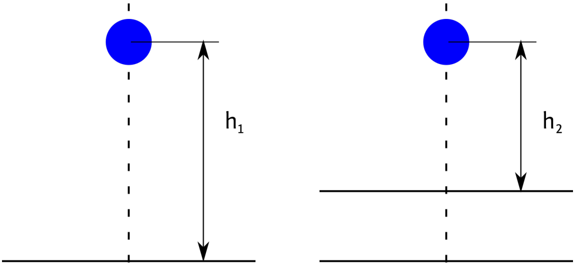

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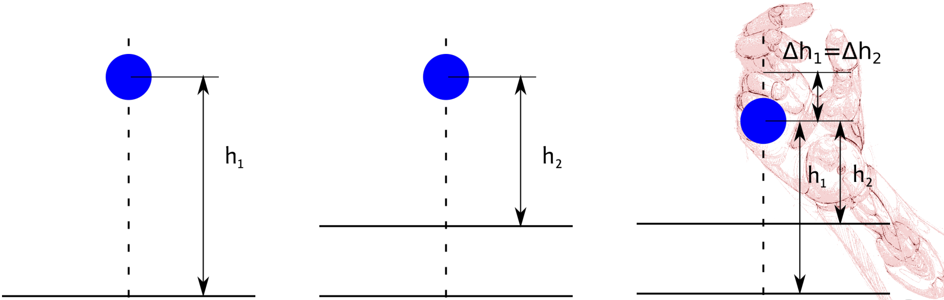

Definitions

Kinetic energy

Kinetic energy

The energy of an object that it possesses due to its motion.

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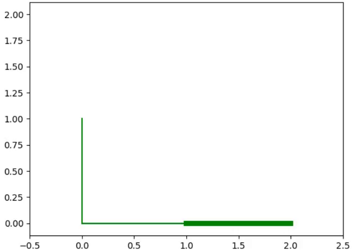
What properties are influencing the kinetic energy of an object.

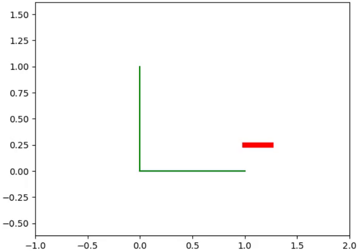
Definitions

Kinetic energy



Kinetic energy due to

linear velocity

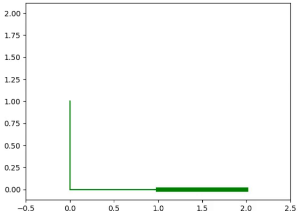


Kinetic energy due to

angular velocity

Definitions

Kinetic energy



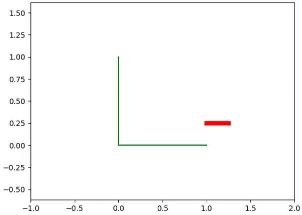
The equation of kinetic energy due to linear velocity is:

$$K_{linear} = \frac{1}{2}\vec{u}^T m \vec{u}$$

Where m is the mass of the object and $\vec{u}$ is the velocity vector.

Definitions

Kinetic energy



The equation of kinetic energy due to angular velocity is:

$$K_{angular} = \frac{1}{2}\vec{\omega}^T I \vec{\omega}$$

Where I is the moment of inertia of the object, and ω is its angular velocity vector.

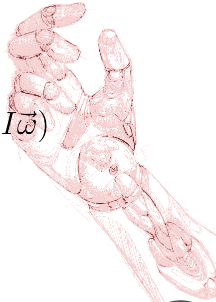
Definitions

Kinetic energy

Total kinetic energy

The total kinetic energy of an object is the sum of its linear and angular kinetic energy.

$$K_{total} = K_{linear} + K_{angular} = \frac{1}{2}(\vec{u}^T m \vec{u} + \vec{\omega}^T I \vec{\omega})$$



Moment of inertia

A tensor that determines the relationship between input torque and angular acceleration about a rotational axis.

The moment of inertia is the equivalent of mass, but for rotational movements.

Definitions

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$$\vec{\tau} = I\dot{\vec{\omega}}$$

Definitions

Moment of inertia

The moment of inertia shows us how 'difficult' is it to rotate an object around an arbitrary axis. It is related with how the mass of the object is distributed in space.

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

This 'difficulty' might be different for the same object, but different axes.

Definitions

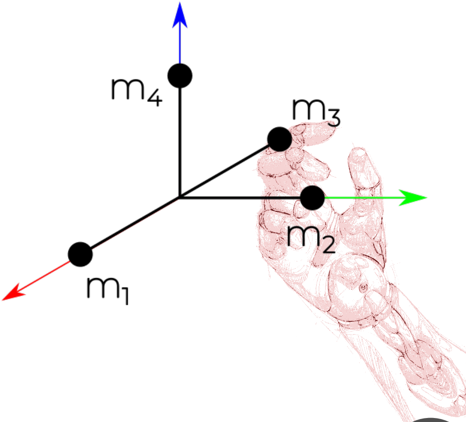
Moment of inertia

How do we calculate it? (for a system of bodies)

$$I_{xx} = \sum_{k=1}^{N} m_k(y_k^2 + z_k^2)$$

$$I_{yy} = \sum_{k=1}^{N} m_k(x_k^2 + z_k^2)$$

$$I_{zz} = \sum_{k=1}^{N} m_k(x_k^2 + y_k^2)$$



Definitions

Moment of inertia

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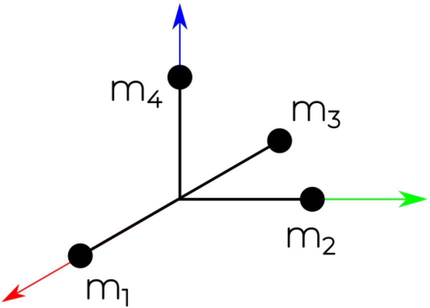

$$I_{xy} = I_{yx} = -\sum_{k=1}^{N} m_k x_k y_k$$

$$I_{xz} = I_{zx} = -\sum_{k=1}^{N} m_k x_k z_k$$

$$I_{yz} = I_{zy} = -\sum_{k=1}^{N} m_k y_k z_k$$

Definitions

Moment of inertia

How do we calculate it? (for a continuous object)

$$I_{xx} = \iiint (y^2 + z^2)\rho(x, y, z)dV \qquad I_{xy} = I_{yx} = -\iiint xy\rho(x, y, z)dV$$

$$I_{yy} = \iiint (x^2 + z^2)\rho(x, y, z)dV \qquad I_{xz} = I_{zx} = -\iiint xz\rho(x, y, z)dV$$

$$I_{zz} = \iiint (x^2 + y^2)\rho(x, y, z)dV \qquad I_{yz} = I_{zy} = -\iiint yz\rho(x, y, z)dV$$

Definitions

Moment of inertia

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For some simple objects, we have analytical solutions (approximations) for the moment of inertia tensor.

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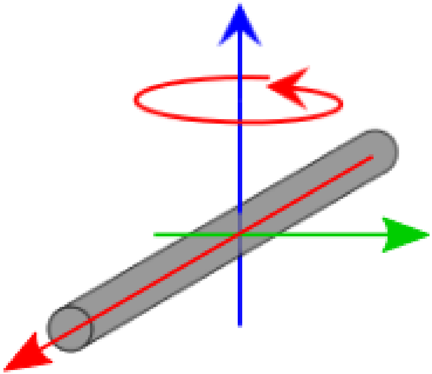


$$I = \begin{bmatrix} \frac{1}{12}ml^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{12}ml^2 \end{bmatrix}$$

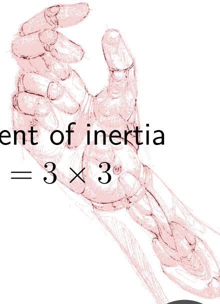
Definitions

Bringing it all together

We define the Lagrangian as the difference between Kinetic and Potential energy of our system

$$L = K - P$$

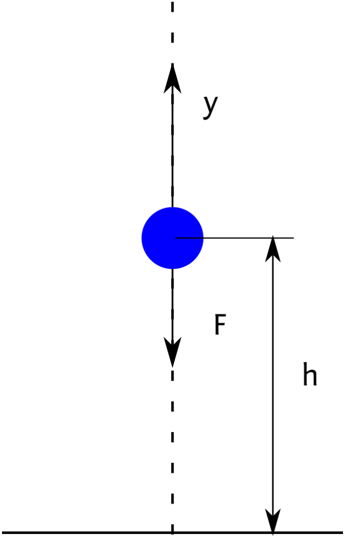
where:



Potential Energy	Kinetic Energy	Moment of inertia
$P = mgh$	$K = \frac{1}{2}(\vec{u}^T m \vec{u} + \vec{\omega}^T I \vec{\omega})$	$I = 3 \times 3$

Lagrangian mechanics

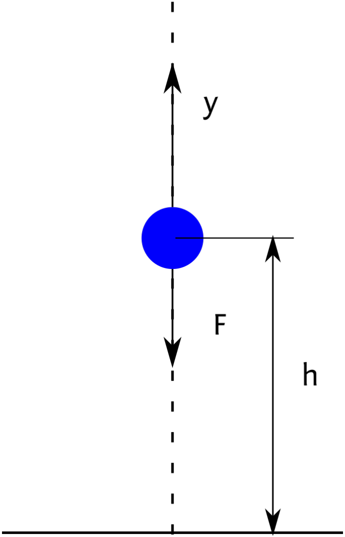
Equivalence with Newton



Let's see if the Lagrangian mechanics are giving us the same results

Lagrangian mechanics

Equivalence with Newton



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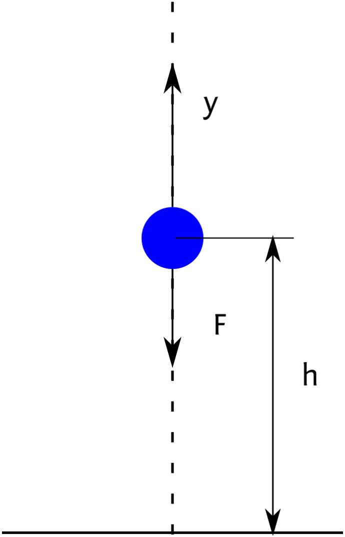
First we need to define the *state* of the system

Lagrangian mechanics

Equivalence with Newton



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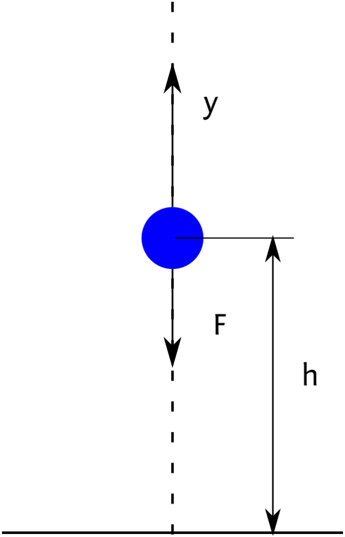
First we need to define the *state* of the system

The state of the system can be fully described by y and $\dot{y}$.

Lagrangian mechanics

Equivalence with Newton



Let's see if the Lagrangian mechanics are giving us the same results

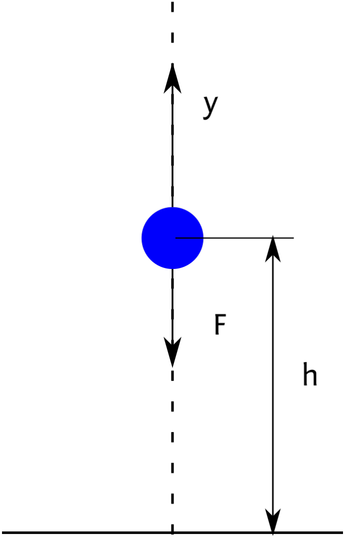
First we need to define the *state* of the system

The state of the system can be fully described by y and $\dot{y}$.

Then we need to write the Lagrangian in terms of the state.

Lagrangian mechanics

Equivalence with Newton



Potential energy:

$$P = mgy$$

Kinetic energy:

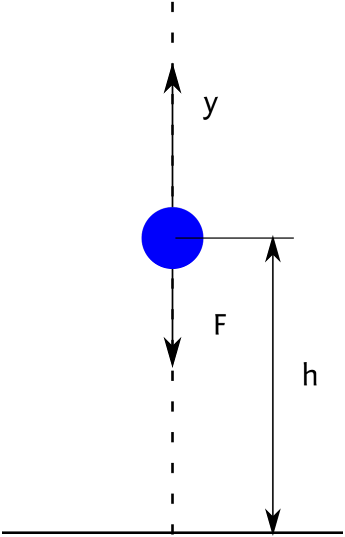
$$K = \frac{1}{2}m\dot{y}^2$$

Therefore:

$$L = \frac{1}{2}m\dot{y}^2 - mgy$$

Lagrangian mechanics

Equivalence with Newton



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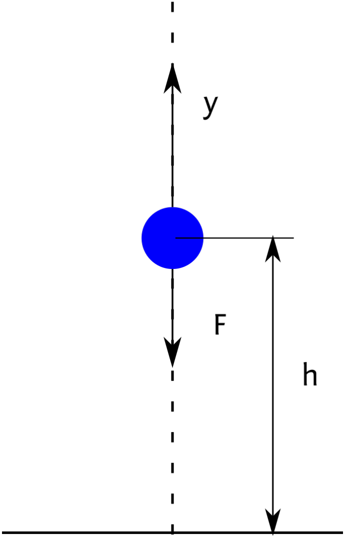
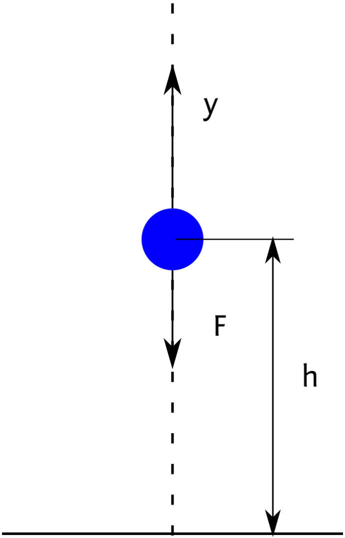


$$L = \frac{1}{2}m\dot{y}^2 - mgy$$

Then we need to diferentiate:

Lagrangian mechanics

Equivalence with Newton



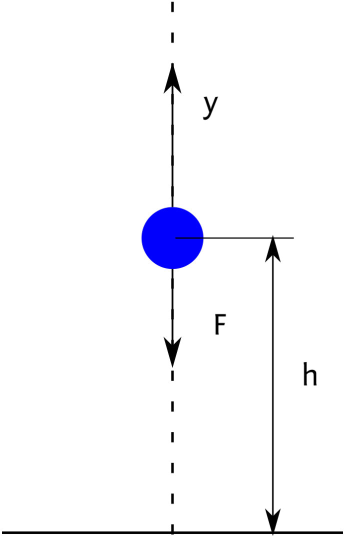
$$L = \frac{1}{2}m\dot{y}^2 - mgy$$

Then we need to diferentiate:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = F$$

Lagrangian mechanics

Equivalence with Newton



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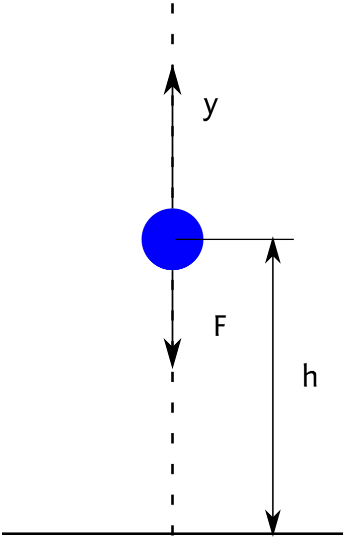
$$\frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

Lagrangian mechanics

Equivalence with Newton



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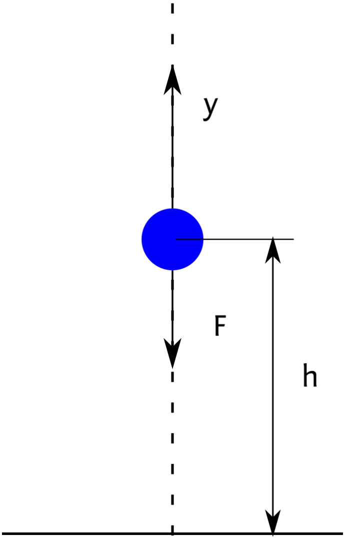
$$\frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = m\ddot{y}$$

Lagrangian mechanics

Equivalence with Newton



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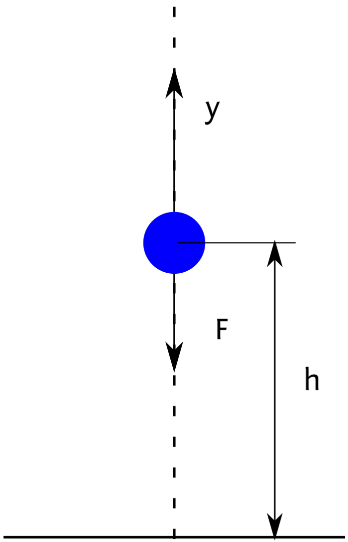
$$\frac{\partial L}{\partial y} = mg$$

Lagrangian mechanics

Equivalence with Newton



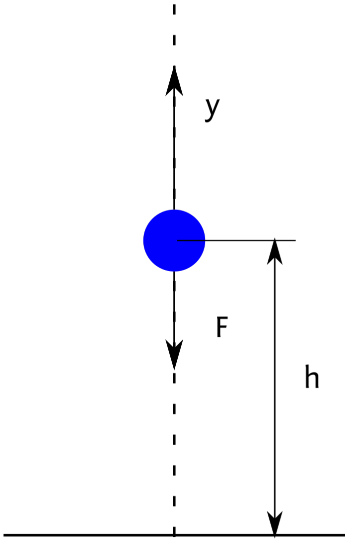
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = F$$

$$m\ddot{y} - mg = F$$

$$F + mg = m\ddot{y}$$

Lagrangian mechanics

Equivalence with Newton



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$$m\ddot{y} - mg = F$$

$$F + mg = m\ddot{y}$$

Same as Newton's second law!

Questions?