

Forward kinematics

Joints, links, degrees of freedom



**TECHNICAL
UNIVERSITY**

OF CLUJ-NAPOCA
ROMANIA

Last update: October 12, 2023

Agenda

- Robot modeling
- Degrees of freedom
- Work envelope
- Joint types
- Links
- Forward kinematics



Recap

What we saw last week

$$Trans(X, a) = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Trans(Y, b) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Trans(Z, c) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Recap

What we saw last week

$$RotX(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$RotY(\phi) = \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$RotZ(\omega) = \begin{bmatrix} \cos\omega & -\sin\omega & 0 & 0 \\ \sin\omega & \cos\omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

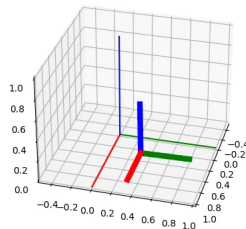


Recap

What we saw last week

Multiplication from the left results in transformation according to the axes of the base coordinate frame.

$$V'_O = RotX(\theta)V_O$$

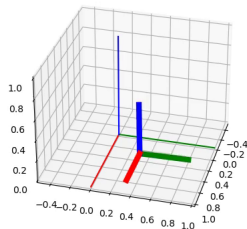


Recap

What we saw last week

Multiplication from the right results in transformation according to the axes of the transformed coordinate frame.

$$V'_O = V_O RotX(\theta)$$

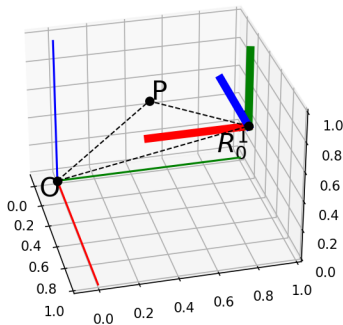


Recap

What we saw last week

Sometimes, we know the coordinates of a point in one coordinate frame, but we need to describe it in a second frame. This is possible if we know the relative position of the two coordinate frames

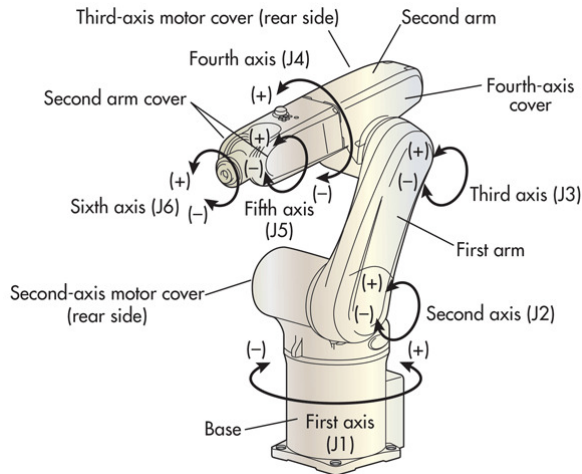
$$P_0 = R_0^1 P_1$$



Robotic modeling

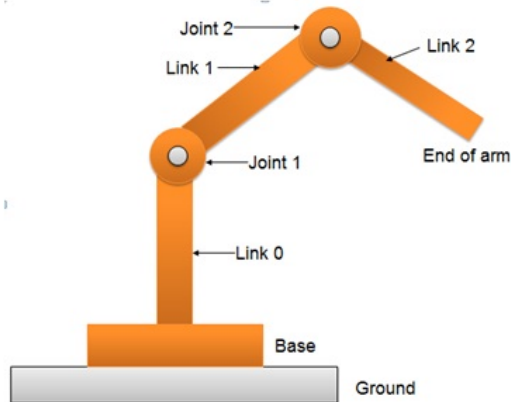
Robotic structure

In this course we will mainly talk about stationary articulated robots



Links

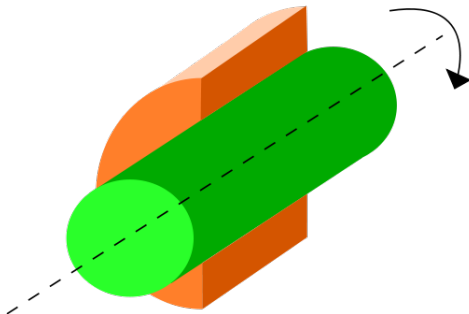
A link is a structural part of any robot. It is usually stiff and is modelled as a rigid (non-deformable) part. They can have any shape, according to the design of the robot.



Joints

Revolute joints

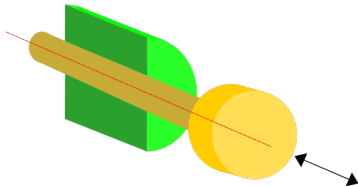
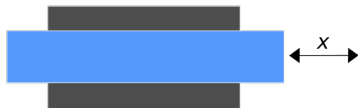
A revolute joint is a joint that allows motion that changes the orientation of a segment by rotating around a fixed axis. They can add one degree of freedom to a robot.



Joints

Prismatic joints

A prismatic joint is a joint that allows motion that changes the position of a segment by translating along an axis. They can add one degree of freedom to a robot.



Robotic modeling

Degrees of freedom

When talking about degrees of freedom for a robot we refer to the freedom of the robot to vary **independently** the position and orientation of its end-effector in different directions.



Robotic modeling

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When talking about degrees of freedom for a robot we refer to the freedom of the robot to vary **independently** the position and orientation of its end-effector in different directions.

E.g. a planar robot with two revolute joints and two links has two degrees of freedom as it can vary **independently** the position of the end-effector in the x and y direction.

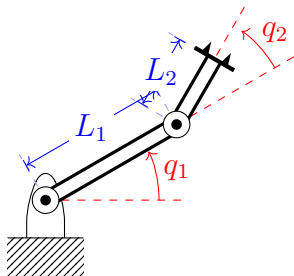


Robotic modeling

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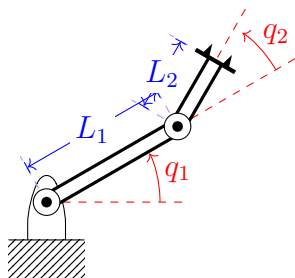


Robotic modeling

Degrees of freedom

When talking about degrees of freedom for a robot we refer to the freedom of the robot to vary **independently** the position and orientation of its end-effector in different directions.

E.g. a planar robot with two revolute joints and two links has two degrees of freedom as it can vary **independently** the position of the end-effector in the x and y direction.

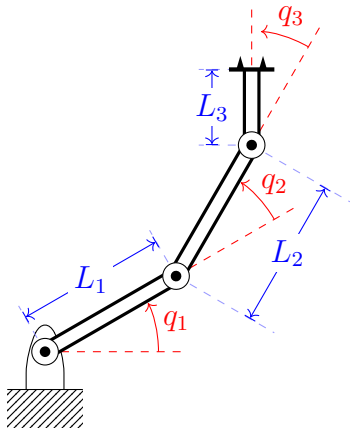


This manipulator has two joints and two degrees of freedom. It can independently vary the x and y position of the end effector. It can also change the x position and orientation of the end effector independently, but then the y position is determined (i.e. is not independent).

Robotic modeling

Degrees of freedom

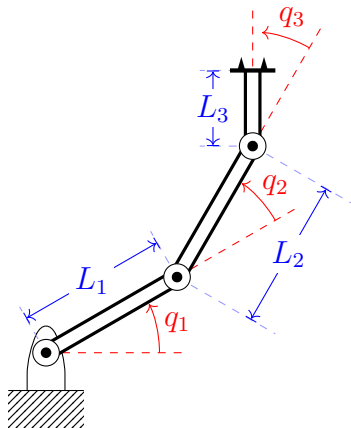
If we add one more joint, we add another degree of freedom



Robotic modeling

Degrees of freedom

If we add one more joint, we add another degree of freedom



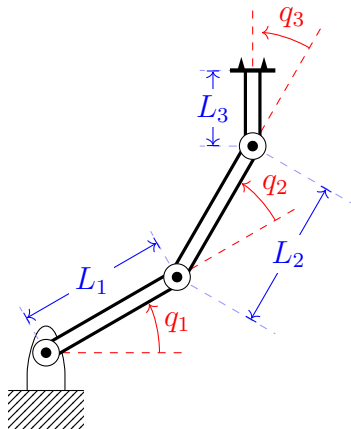
This manipulator can independently vary the x and y position of the end effector, while also controlling the orientation (3 degrees of freedom).



Robotic modeling

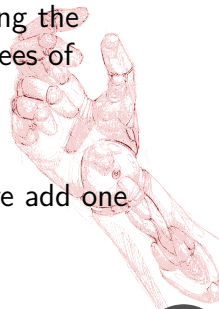
Degrees of freedom

If we add one more joint, we add another degree of freedom



This manipulator can independently vary the x and y position of the end effector, while also controlling the orientation (3 degrees of freedom).

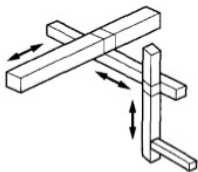
What happens if we add one more joint?



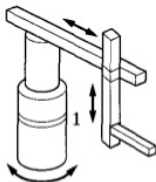
Robotic modeling

Work envelope

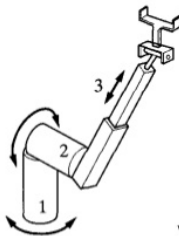
The combination of links and joints defines the degrees of freedom to a robot. Besides that, it also defines the work envelope of the robot.



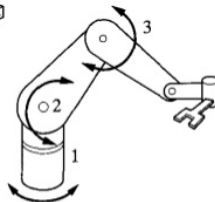
Cartesian



Cylindrical



Spherical



Articulated



Robotic modeling

Work envelope

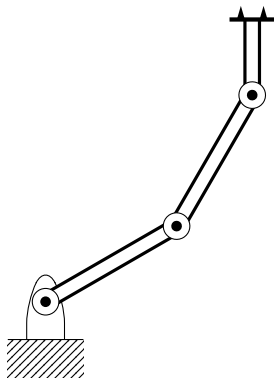
Even if a robot has a certain number of degrees of freedom, some of them might be lost in some areas of the work envelope.



Robotic modeling

Work envelope

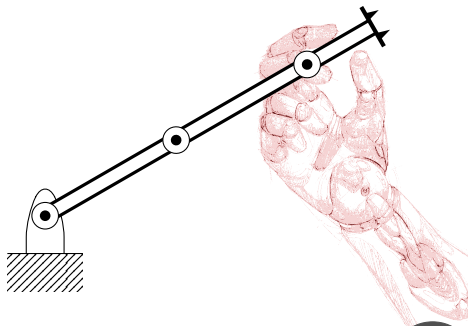
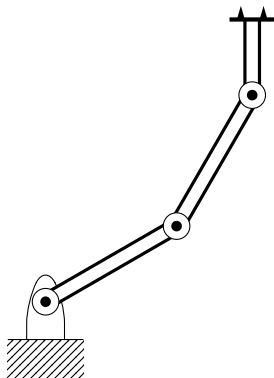
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Robotic modeling

Work envelope

Even if a robot has a certain number of degrees of freedom, some of them might be lost in some areas of the work envelope.



Forward kinematics

Definition

The forward kinematics model (FKM) is a mathematical tool that allows us to calculate the position and orientation (pose) of a robot's point of interest if we know the state of the joints and the lengths of the links.



Forward kinematics

Definition

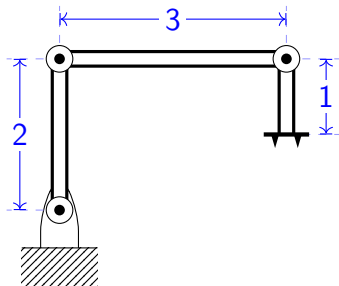
The forward kinematics model (FKM) is a mathematical tool that allows us to calculate the position and orientation (pose) of a robot's point of interest if we know the state of the joints and the lengths of the links.

In simple words

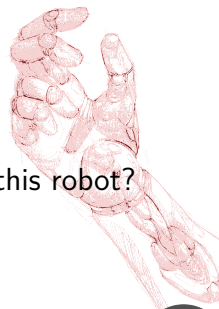
How do I calculate the pose of the end-effector if I know the joint angles (coordinates)?

Forward kinematics

Definition



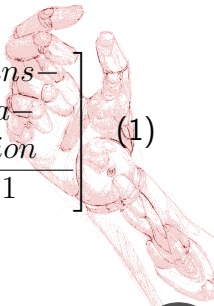
How do we calculate the pose of the end-effector of this robot?



Forward kinematics

Definition

We describe the pose of the end-effector using a 4x4 transformation matrix (contains information about position and orientation).

$$T = \left[\begin{array}{ccc|c} 3 & \times & 3 & 3 \times 1 \\ \hline 1 & \times & 3 & 1 \times 1 \end{array} \right] = \left[\begin{array}{ccc|c} & & & \text{translation} \\ & & & \\ & & & \\ \hline 0 & 0 & 0 & 1 \end{array} \right] \quad (1)$$


Forward kinematics

Calculation

To define the forward kinematics model we perform the following steps:



Forward kinematics

Calculation

To define the forward kinematics model we perform the following steps:

- We identify the links and joints of the robot.

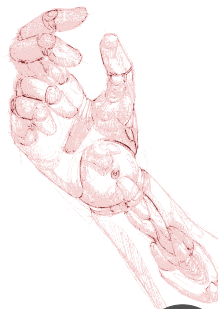


Forward kinematics

Calculation

To define the forward kinematics model we perform the following steps:

- We identify the links and joints of the robot.
- We attach a fixed coordinate frame on the base of the robot.



Forward kinematics

Calculation

To define the forward kinematics model we perform the following steps:

- We identify the links and joints of the robot.
- We attach a fixed coordinate frame on the base of the robot.
- We attach a coordinate frame on each link at their joints.



Forward kinematics

Calculation

To define the forward kinematics model we perform the following steps:

- We identify the links and joints of the robot.
- We attach a fixed coordinate frame on the base of the robot.
- We attach a coordinate frame on each link at their joints.
- We attach a coordinate frame at the end-effector.

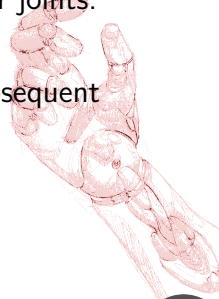


Forward kinematics

Calculation

To define the forward kinematics model we perform the following steps:

- We identify the links and joints of the robot.
- We attach a fixed coordinate frame on the base of the robot.
- We attach a coordinate frame on each link at their joints.
- We attach a coordinate frame at the end-effector.
- We calculate the transformation between each subsequent coordinate frame.

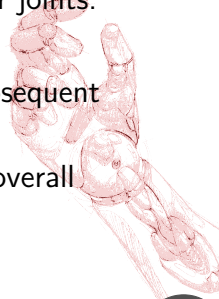


Forward kinematics

Calculation

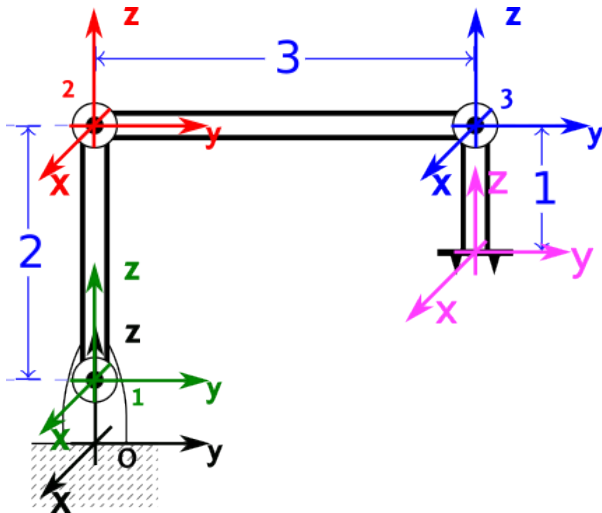
To define the forward kinematics model we perform the following steps:

- We identify the links and joints of the robot.
- We attach a fixed coordinate frame on the base of the robot.
- We attach a coordinate frame on each link at their joints.
- We attach a coordinate frame at the end-effector.
- We calculate the transformation between each subsequent coordinate frame.
- We combine the transformations to calculate the overall transformation from base to end-effector.



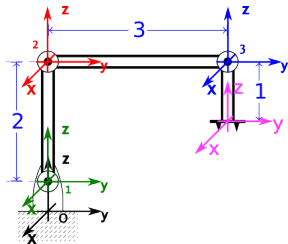
Forward kinematics

Calculation



Forward kinematics

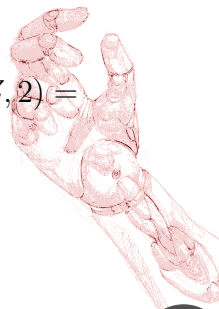
Static calculation



$$R_0^1 = Trans(Z, 1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

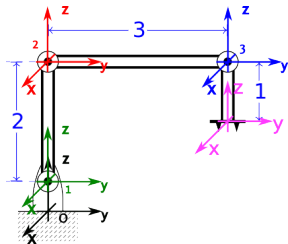
$$R_1^2 = Trans(Z, 2) =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Forward kinematics

Static calculation

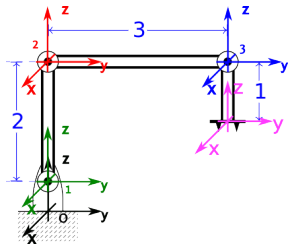


$$R_2^3 = Trans(Y, 3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_3^4 =$$



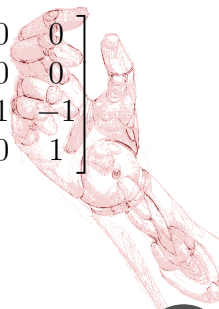
Forward kinematics

Static calculation



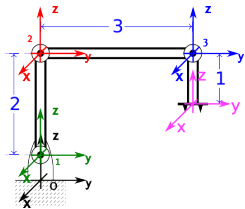
$$R_2^3 = Trans(Y, 3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_3^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



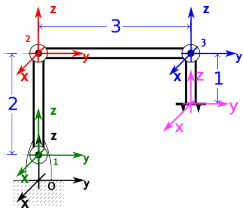
Forward kinematics

Static calculation



Forward kinematics

Static calculation

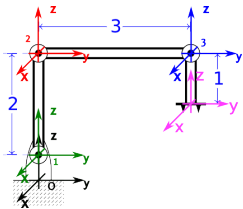


$$R_0^4 =$$



Forward kinematics

Static calculation

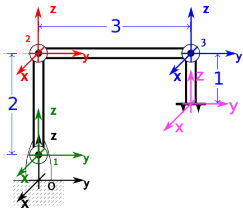


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Forward kinematics

Static calculation

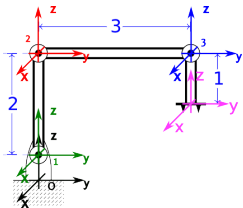


$$R_0^4 = R_0^1 R_1^2 R_2^3 R_3^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Forward kinematics

Static calculation

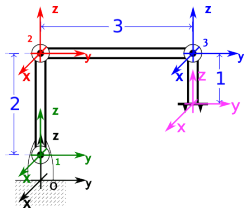


$$R_0^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & & & \text{translation} \\ \text{rotation} & & & \\ & & & \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Forward kinematics

Static calculation



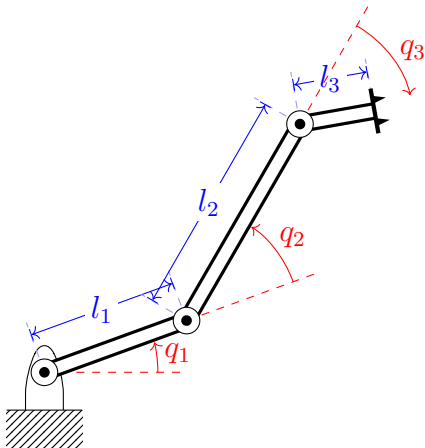
$$R_0^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & & & \text{translation} \\ & \text{rotation} & & \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_m^n = \begin{bmatrix} X_x & Y_x & Z_x & P_x \\ X_y & Y_y & Z_y & P_y \\ X_z & Y_z & Z_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



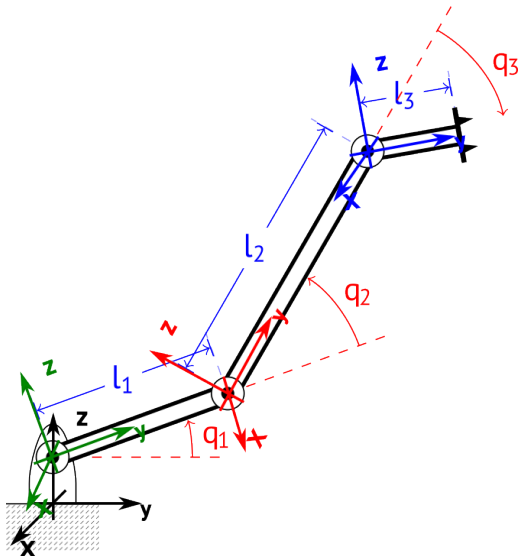
Forward kinematics

What about other configurations?



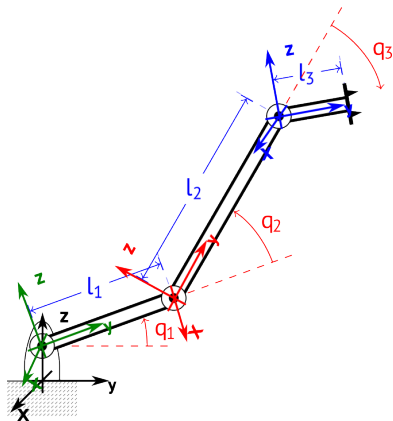
Forward kinematics

Dynamic calculation



Forward kinematics

Dynamic calculation

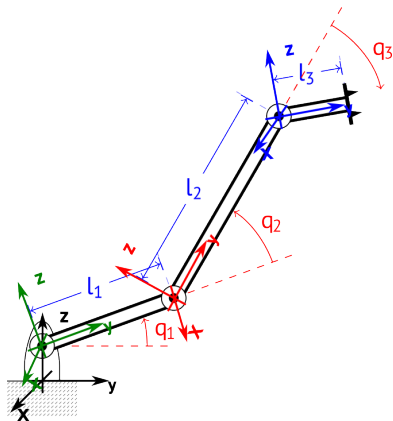


$$R_0^1 = Trans(Z, l_1)R(X, q_1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_1 & -s_1 & 0 \\ 0 & s_1 & c_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1^2 = Trans(Y, l_2)R(X, q_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_2 & -s_2 & 0 \\ 0 & s_2 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward kinematics

Dynamic calculation



$$R_2^3 = Trans(Y, l_2)R(X, q_3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_3 & -s_3 & 0 \\ 0 & s_3 & c_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

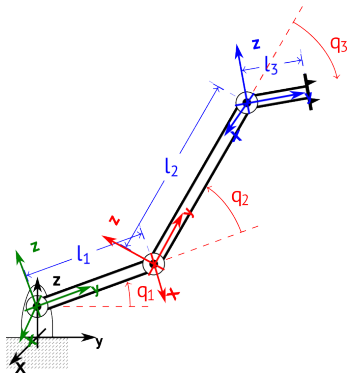
$$R_3^4 = Trans(Y, l_3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$R_0^4 = R_0^1 R_1^2 R_2^3 R_3^4$$

Forward kinematics

Dynamic calculation



$$R_0^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{1,2,3} & -s_{1,2,3} & l_3 c_{1,2,3} + l_2 c_{1,2} + l_1 c_1 \\ 0 & s_{1,2,3} & c_{1,2,3} & l_3 s_{1,2,3} + l_2 s_{1,2} + l_1 s_1 + 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Forward kinematics

Dynamic calculation

$$R_0^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{1,2,3} & -s_{1,2,3} & l_3 c_{1,2,3} + l_2 c_{1,2} + l_1 c_1 \\ 0 & s_{1,2,3} & c_{1,2,3} & l_3 s_{1,2,3} + l_2 s_{1,2} + l_1 s_1 + 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward kinematics

The FKM is a transformation matrix, a function of the joint positions and link lengths. If we know these variables, we can calculate the position and orientation of the end effector (or any other point).

Forward kinematics

Food for thought

Why do we multiply from the right each transformation?



Forward kinematics

Food for thought

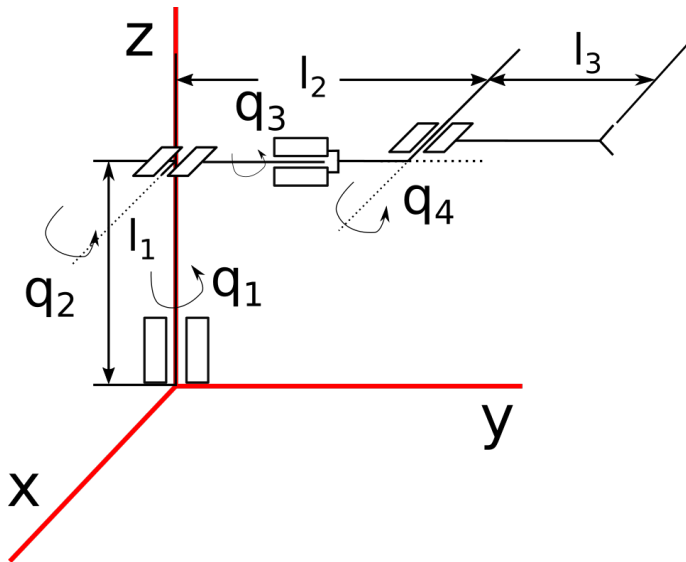
Why do we multiply from the right each transformation?

Could we figure out how to multiply from the left?



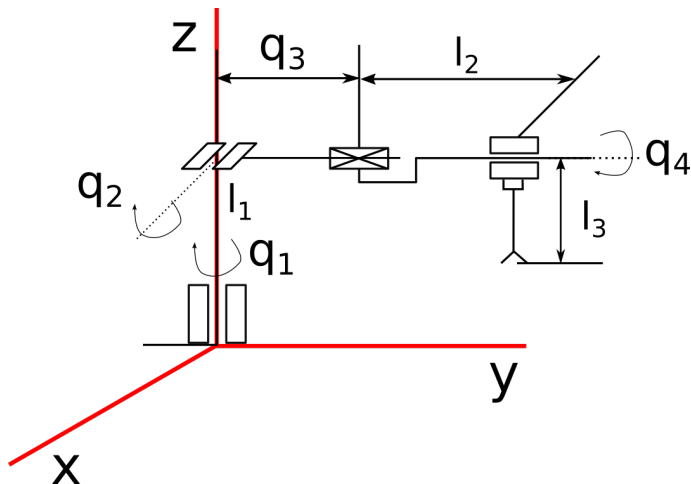
Forward kinematics

Example in 3D



Forward kinematics

Example in 3D





Questions?